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# Analytical Bethe Ansatz for closed and open $gl(\mathcal{M}|\mathcal{N})$ super-spin chains in arbitrary representations and for any Dynkin diagram

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## Abstract

We present the analytical Bethe ansatz for spin chains based on the superalgebras  $gl(\mathcal{M}|\mathcal{N})$ ,  $\mathcal{M} \neq \mathcal{N}$ , with at each site an arbitrary representation (and including inhomogeneities). The calculation is done for closed and open spin chains. In this latter case, the boundary matrices  $K_{\pm}(\lambda)$  are of general type, provided they commute. We compute the Bethe ansatz equations in full generality, and for any type of Dynkin diagram. Examples are worked out to illustrate the techniques.

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# 1 Introduction

The possibility of constructing and solving by algebraic and/or analytical methods one-dimensional interacting quantum spin chains, is one of the major achievements of quantum integrable systems. It allows the determination of the spectrum, eigenvectors and (at least partially) the calculation of correlation functions. The main tool is the quantum  $R$ -matrix, obeying a cubic Yang-Baxter equation, the “coproduct” properties of which allow the building of a periodic  $L$ -site transfer matrix with identical exchange relations and the subsequent derivation of quantum commuting Hamiltonians [1]. A similar structure arises for non-periodic (open) spin chains. These are characterised by a second object: the reflection matrix  $K$ , obeying a quadratic consistency equation with the  $R$  matrix [2–6]. Using again “coproduct-like” properties of this structure one constructs suitable transfer matrices yielding (local) commuting spin chain Hamiltonians by combining  $K$  and semi-tensor products of  $R$  [3].

Recently, a more algebraic approach to the analytical Bethe ansatz has been developed, allowing a ‘universal’ approach (i.e; whatever the spins on the chain) to the spectrum of the transfer matrix, and the corresponding Bethe equations. This framework has been developed for open and closed spin chains, based on  $gl(\mathcal{N})$  [7] and  $\mathcal{U}_q(gl_N)$  [8] algebras.

On an other hand, quantum supersymmetric integrable systems appeared [9] in the context of  $N = 4$  super-Yang-Mills (SYM) theories, in the loop expansion of the dilatation operator, used for the computation of anomalous dimensions of trace operators. In fact, it seems that (at least for the first loop corrections) that the dilatation operator can be identified with some super-spin chain Hamiltonian, the type of the chain depending both on the (sub)sector of the SYM theory one considers, and on the order of loop correction, see e.g. [10].

Hence, it is the right time to give a general overview of the possible integrable closed and open super-spin chains that one can construct starting from a  $gl(\mathcal{M}|\mathcal{N})$  superalgebra and arbitrary spins on the chain. We will study the spectrum and Bethe equations associated to these chains. Closed spin chains based on  $sl(\mathcal{M}|\mathcal{N})$  superalgebras in the distinguished diagram were studied in [11] and [12] and, in the case of alternating fundamental-conjugate representations of  $sl(\mathcal{M}|\mathcal{N})$  in [13]. In [14], closed spin chains in the fundamental representation but for any type of Dynkin diagram were studied using the Baxter  $Q$ -operator, and generalized in [15] to a chain where all the spins are in a (type 1) typical representation depending on a free parameter. General approach using Hirota equation was done in [16]. Open spin chains based on  $sl(1|2)$  have been studied in details in e.g. [17, 18]. The  $sl(\mathcal{M}|\mathcal{N})$  case with spins in the fundamental representation, with diagonal  $K(u)$  matrices, but for any type of Dynkin diagrams have been done in [26]. The deformed case for fundamental representations but general  $K(u)$  matrices have been studied in [19]. We will use the algebro-analytical framework developed in [7, 8], applied to superalgebras. It will provide a ‘universal’ presentation for all chains (whatever the representations that enter the chain), for closed and open cases. A particularity of superalgebras (that do not share usual algebras) is the existence of different Dynkin diagrams for the same superalgebra. This leads to different presentations of the spectrum of the same transfer matrix, hence to different Bethe equations: the presentation is also universal in the sense that it applies for all Dynkin diagrams of the superalgebra.

The plan of the paper is as follows. In section 2, we present the algebraic structures that are needed for the construction of super-spin chains: the super-Yangian based on  $gl(\mathcal{M}|\mathcal{N})$  for closed chains and the reflection superalgebra for open chains. Then, in section 3, we construct

the closed spin chains, give their spectrum and their Bethe equations, in the case of distinguished Dynkin diagram. Section 4 is devoted to the general form of the Bethe equations for each of the different Dynkin diagrams of the superalgebra. The case of open super-spin chain is treated in section 5, including the different presentations associated to different Dynkin diagrams. Finally, section 6 illustrates our method on examples.

## 2 Algebraic structures

### 2.1 Graded spaces

We will work on  $\mathbb{Z}_2$ -graded spaces  $\mathbb{C}^{\mathcal{M}|\mathcal{N}}$ , with  $\mathbb{Z}_2$ -grade

$$[\ ] : \begin{cases} \mathbb{N}_{\mathcal{M}+\mathcal{N}} & \rightarrow \{0, 1\} \\ j & \mapsto [j] \end{cases} \quad (2.1)$$

where  $\mathbb{N}_{\mathcal{M}+\mathcal{N}} = \{1, 2, \dots, \mathcal{M} + \mathcal{N}\}$ . The elementary  $\mathbb{C}^{\mathcal{M}|\mathcal{N}}$  vectors  $e_i$  and  $End(\mathbb{C}^{\mathcal{M}|\mathcal{N}})$  matrices  $E_{ij}$  have grade

$$[e_i] = [i] \quad \text{and} \quad [E_{ij}] = [i] + [j]. \quad (2.2)$$

The tensor product is graded accordingly:

$$(E_{ij} \otimes E_{kl})(E_{ab} \otimes E_{cd}) = (-1)^{([k]+[l])([a]+[b])}(E_{ij}E_{ab} \otimes E_{kl}E_{cd}). \quad (2.3)$$

The permutation operator

$$P_{12} = \sum_{i,j=1}^{\mathcal{M}+\mathcal{N}} (-1)^{[j]} E_{ij} \otimes E_{ji} \quad (2.4)$$

is also graded

$$P_{12}(e_i \otimes e_j) = (-1)^{[i][j]} e_j \otimes e_i \quad \text{and} \quad P_{12}(E_{ij} \otimes E_{kl})P_{12} = (-1)^{([i]+[j])([k]+[l])} E_{kl} \otimes E_{ij}. \quad (2.5)$$

The permutation operator obey the relation  $P_{12}^2 = \mathbb{I} \otimes \mathbb{I}$ , so that it is symmetric:

$$P_{21} = P_{12} P_{12} P_{12} = P_{12} \quad (2.6)$$

Together with the  $\mathbb{Z}_2$ -grading, we will use a graded commutator  $[\cdot, \cdot]$ , which is graded antisymmetric and obeys a graded Jacobi identity.

Unless explicitly specified, we will work with the *distinguished*  $\mathbb{Z}_2$ -grade defined by

$$[i] = \begin{cases} 0, & 1 \leq i \leq \mathcal{M}, \\ 1, & \mathcal{M} + 1 \leq i \leq \mathcal{M} + \mathcal{N}. \end{cases} \quad (2.7)$$

However, in some cases, we will use different grading, such as the *symmetric*  $\mathbb{Z}_2$ -grade, defined for  $\mathcal{N} = 2n$ :

$$[i] = \begin{cases} 0, & 1 \leq i \leq n \quad \text{and} \quad \mathcal{M} + n + 1 \leq i \leq \mathcal{M} + \mathcal{N}, \\ 1, & n + 1 \leq i \leq \mathcal{M} + n. \end{cases} \quad (2.8)$$

The name of these grading refers to the  $gl(\mathcal{M}|\mathcal{N})$  Dynkin diagram (and simple roots) they are associated to, see below.

## 2.2 The $gl(\mathcal{M}|\mathcal{N})$ superalgebra

The Lie superalgebra  $gl(\mathcal{M}|\mathcal{N})$  is a  $\mathbb{Z}_2$ -graded vector space over  $\mathbb{C}$  spanned by the basis  $\{\mathcal{E}_{ab}|a, b = 1, 2, \dots, \mathcal{M} + \mathcal{N}\}$ . The gradation is defined by the  $\mathbb{Z}_2$ -grade [ ] through:

$$[\mathcal{E}_{ab}] = [a] + [b]. \quad (2.9)$$

The bilinear graded commutator associated to  $gl(\mathcal{M}|\mathcal{N})$  is defined by:

$$\{\mathcal{E}_{ab}, \mathcal{E}_{cd}\} = \delta_{cb} \mathcal{E}_{ad} - (-1)^{([a]+[b])([c]+[d])} \delta_{ad} \mathcal{E}_{cb}. \quad (2.10)$$

Gathering the generators  $\mathcal{E}_{ab}$  into a single matrix

$$\mathbb{E} = \sum_{a,b=1}^{\mathcal{M}+\mathcal{N}} (-1)^{[a]} \mathcal{E}_{ab} E_{ab} \quad (2.11)$$

the above commutation relations can be recasted as

$$[\mathbb{E}_1, \mathbb{E}_2] = P_{12}(\mathbb{E}_2 - \mathbb{E}_1) \quad (2.12)$$

where  $\mathbb{E}_1 = \mathbb{E} \otimes \mathbb{I}$  and  $\mathbb{E}_2 = \mathbb{I} \otimes \mathbb{E}$ .

Although the  $gl(\mathcal{M}|\mathcal{N})$  superalgebra is a graded version of the  $gl(\mathcal{M} + \mathcal{N})$  algebra, they differ on several points, a common feature when comparing Lie algebras and superalgebras, see e.g. [20] for more details. In particular, there exist several inequivalent simple roots systems, leading to different presentations of the same superalgebra. One can relate these different systems to a choice of the  $\mathbb{Z}_2$ -grade. To each inequivalent simple roots system correspond a Dynkin diagram, so that a superalgebra possesses several Dynkin diagram. Note however that any Dynkin diagram defines uniquely a superalgebra.

## 2.3 The super-Yangian $\mathcal{Y}(\mathcal{M}|\mathcal{N})$

$\mathcal{Y}(\mathcal{M}|\mathcal{N})$  is the graded unital associative algebra, with generators  $T_{ab}^{(n)}$ ,  $n > 0$ ,  $a, b = 1, \dots, \mathcal{M} + \mathcal{N}$ , with  $\mathbb{Z}_2$ -grade

$$[T_{ab}^{(n)}] = [a] + [b], \quad \forall a, b, n. \quad (2.13)$$

We gather  $\mathcal{Y}(\mathcal{M}|\mathcal{N})$  generators in matrix form with  $T_{ab}^{(0)} = \delta_{ab}$

$$T(u) \doteq \sum_{a,b=1}^{\mathcal{M}+\mathcal{N}} \sum_{n \geq 0} \frac{\hbar^n}{u^n} T_{ab}^{(n)} E_{ab} \doteq \sum_{n \geq 0} \frac{\hbar^n}{u^n} T^{(n)} \doteq \sum_{a,b}^{\mathcal{M}+\mathcal{N}} T_{ab}(u) E_{ab}, \quad (2.14)$$

which is an even element of  $\mathcal{Y}(\mathcal{M}|\mathcal{N})[u^{-1}] \otimes \text{End}(\mathbb{C}^{\mathcal{M}|\mathcal{N}})$ . Here and below, the space  $\text{End}(\mathbb{C}^{\mathcal{M}|\mathcal{N}})$  will be referred as the auxiliary space, while (the copies of) the super-Yangian  $\mathcal{Y}(\mathcal{M}|\mathcal{N})[u^{-1}]$  will be called the quantum space(s).

$\mathcal{Y}(\mathcal{M}|\mathcal{N})$  commutation relations are given by the so-called FRT exchange relation [21]

$$R_{12}(u - v) T_1(u) T_2(v) = T_2(v) T_1(u) R_{12}(u - v), \quad (2.15)$$

each side of the equation being an element of  $\mathcal{Y}(\mathcal{M}|\mathcal{N})[u^{-1}] \otimes \text{End}(\mathbb{C}^{\mathcal{M}|\mathcal{N}}) \otimes \text{End}(\mathbb{C}^{\mathcal{M}|\mathcal{N}})$ , and where we have introduced the super-Yangian  $R$ -matrix<sup>1</sup>

$$R_{12}(u) = u \mathbb{I}_{\mathcal{M}+\mathcal{N}} \otimes \mathbb{I}_{\mathcal{M}+\mathcal{N}} - \hbar P_{12}. \quad (2.16)$$

It acts on the two auxiliary spaces associated to  $T_1(u) = T(u) \otimes \mathbb{I}_{\mathcal{M}+\mathcal{N}}$  and  $T_2(u) = \mathbb{I}_{\mathcal{M}+\mathcal{N}} \otimes T(u)$ . The deformation parameter  $\hbar$  is in fact irrelevant (provided it is not zero), hence it is in general set to 1 for algebraic studies. However, in the context of spin chain models, it is set to  $-i$ , so that we keep it free to encompass these two choices. Note that the  $R$ -matrix is a globally even one. Its inverse reads

$$R_{12}^{-1}(x) = \frac{1}{x^2 - \hbar^2} (x \mathbb{I} \otimes \mathbb{I} + \hbar P_{12}) = \frac{-1}{x^2 - \hbar^2} R_{12}(-x). \quad (2.17)$$

Projecting the relation (2.15) on elementary matrices  $E_{ab} \otimes E_{cd}$ , one gets

$$\left[ T_{ab}(u), T_{cd}(v) \right] = \frac{(-1)^{\eta(a,b,c)} \hbar}{u - v} \left( T_{cb}(u) T_{ad}(v) - T_{cb}(v) T_{ad}(u) \right), \quad (2.18)$$

where  $\eta(a, b, c) = [a]([b] + [c]) + [b][c]$  and  $[\cdot, \cdot]$  denotes the supercommutator.

Expanding the commutation relation in  $u^{-1}$  and  $v^{-1}$ , we obtain

$$\left[ T_{ab}^{(m)}, T_{cd}^{(n)} \right] = (-1)^{\eta(a,b,c)} \sum_{p=0}^{\min(m,n)-1} \left( T_{cb}^{(p)} T_{ad}^{(m+n-1-p)} - T_{cb}^{(m+n-1-p)} T_{ad}^{(p)} \right), \quad (2.19)$$

This commutation relation shows that the generators  $(-1)^{[a]} T_{ab}^{(1)}$  span a  $gl(\mathcal{M}|\mathcal{N})$  sub-superalgebra of the super-Yangian. Conversely, one can construct a morphism from the Lie superalgebra to the super-Yangian, called the evaluation map:

$$ev : \begin{cases} gl(\mathcal{M}|\mathcal{N}) & \rightarrow \mathcal{Y}(\mathcal{M}|\mathcal{N}) \\ T_{ab}(u) & \mapsto \delta_{ab} + \frac{\hbar}{u} (-1)^{[a]} \mathcal{E}_{ba} \\ T(u) & \mapsto \mathbb{I} + \frac{\hbar}{u} \mathbb{E} \end{cases} \quad (2.20)$$

Using the commutation relations (2.12) of  $gl(\mathcal{M}|\mathcal{N})$ , it is easy to show that  $ev(T(u))$  obey the relation (2.15).

Two subalgebras of  $\mathcal{Y}(\mathcal{M}|\mathcal{N})$  will be used in the following: the Yangian  $\mathcal{Y}(\mathcal{M})$ , generated by  $\{T_{ab}(u), [a] = [b] = 0\}$  and the Yangian  $\mathcal{Y}_{-\hbar}(\mathcal{N})$ , generated by  $\{T_{ab}(u), [a] = [b] = 1\}$ . The generators of these subalgebras are obtained from  $T(u)$  using suitable  $\text{End}(\mathbb{C}^{\mathcal{M}|\mathcal{N}})$  projectors:

$$T^{(\mathcal{M})}(u) = \mathbb{I}_{\mathcal{M}} T(u) \mathbb{I}_{\mathcal{M}} \quad \text{with} \quad \mathbb{I}_{\mathcal{M}} = \sum_{i, [i]=1} E_{ii},$$

$$T_{-\hbar}^{(\mathcal{N})}(u) = \mathbb{I}_{\mathcal{N}} T(u) \mathbb{I}_{\mathcal{N}} \quad \text{with} \quad \mathbb{I}_{\mathcal{N}} = \sum_{i, [i]=0} E_{ii}.$$

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<sup>1</sup>The normalization is chosen in such a way that  $R(u)$  is analytic in  $u$ .

The map

$$\Delta : \begin{cases} \mathcal{Y}(\mathcal{M}|\mathcal{N})[u^{-1}] & \rightarrow \mathcal{Y}(\mathcal{M}|\mathcal{N})[u^{-1}] \otimes \mathcal{Y}(\mathcal{M}|\mathcal{N})[u^{-1}] \\ T_{ij}(u) & \mapsto \Delta(T_{ij}(u)) = \sum_{k=1}^{\mathcal{M}+\mathcal{N}} T_{ik}(u) \otimes T_{kj}(u) \end{cases} \quad (2.21)$$

is an homomorphism of  $\mathcal{Y}(\mathcal{M}|\mathcal{N})$ . Gathering the generators into matrices, it rewrites

$$\Delta(T(u)) = T(u) \dot{\otimes} T(u) \in \mathcal{Y}(\mathcal{M}|\mathcal{N})[u^{-1}] \otimes \mathcal{Y}(\mathcal{M}|\mathcal{N})[u^{-1}] \otimes \text{End}(\mathbb{C}^{\mathcal{M}|\mathcal{N}}). \quad (2.22)$$

$\Delta$  is coassociative:

$$\Delta^{(n)} = (\Delta^{(n-1)} \otimes \text{id}) \Delta = (\text{id} \otimes \Delta^{(n-1)}) \Delta. \quad (2.23)$$

### 2.3.1 Highest weight vectors and modules

A  $\mathcal{Y}(\mathcal{M}|\mathcal{N})$  module  $V$  is said to be highest weight if there exists  $v \in V$  such that

$$\begin{cases} T_{aa}(u)v = \lambda_a(u)v, & \lambda_a(u) \in 1 + u^{-1}\mathbb{C}[u^{-1}] \quad \forall a = 1, \dots, \mathcal{M} + \mathcal{N} \\ T_{ab}(u)v = 0, & 1 \leq b < a \leq \mathcal{M} + \mathcal{N} \end{cases} \quad (2.24)$$

The vector  $\lambda(u) \doteq (\lambda_1(u), \dots, \lambda_{\mathcal{M}+\mathcal{N}}(u))$  is the highest weight of  $V$ , and  $v$  a highest weight vector. The following theorems have been proved in [22]

**Theorem 2.1** *Any finite-dimensional irreducible representation of  $\mathcal{Y}(\mathcal{M}|\mathcal{N})$  admits a unique highest weight vector (up to normalization).*

**Theorem 2.2** *An irreducible representation with highest weight  $\lambda(u)$  is finite-dimensional if and only if*

$$\frac{\lambda_a(u)}{\lambda_{a+1}(u)} = \frac{P_a(u + \hbar)}{P_a(u)}, \quad 1 \leq a \leq \mathcal{M} + \mathcal{N} \quad \text{and} \quad a \neq \mathcal{M}, \quad \frac{\lambda_{\mathcal{M}}(u)}{\lambda_{\mathcal{M}+1}(u)} = \frac{P_{\mathcal{M}}(u)}{P_{\mathcal{M}+\mathcal{N}}(u)}, \quad (2.25)$$

where all  $P_a(u)$  are monic polynomials.

Among the finite-dimensional highest weight representations, there is a class of particular interest, constructed from the evaluation map: an evaluation representation  $ev_{\pi_\mu} = \pi_\mu \circ ev$  is a morphism from the super-Yangian  $Y(\mathcal{M}|\mathcal{N})$  to a highest weight irreducible representation  $\pi_\mu$  of  $gl(\mathcal{M}|\mathcal{N})$ . The morphism is given by:

$$ev_{\pi_\mu}(T_{ij}(u)) = \delta_{ij} + (-1)^{[i]} \pi_\mu(\mathcal{E}_{ji}) \frac{\hbar}{u - a} \quad \forall i, j \in \{1, \dots, \mathcal{M} + \mathcal{N}\}, \quad a \in \mathbb{C}, \quad (2.26)$$

where the dependance (that will be left implicit in what follows) of  $ev_{\pi_\mu}$  on an arbitrary complex shift of the spectral parameter has been introduced. One has

$$ev_{\pi_\mu}(T_{ij}^{(1)}) = (-1)^{[i]} \pi_\mu(\mathcal{E}_{ji}) ; \quad ev_{\pi_\mu}(T_{ij}^{(r)}) = 0 \quad \text{for} \quad r > 1. \quad (2.27)$$

The highest weight  $\mu(u) = (\mu_1(u), \dots, \mu_{\mathcal{M}+\mathcal{N}}(u))$  of the representation  $ev_{\pi_\mu}$  is given by:

$$\mu_i(u) = 1 + (-1)^{[i]} \mu_i \frac{\hbar}{u - a}, \quad \forall i \in \{1, \dots, \mathcal{M} + \mathcal{N}\} \quad (2.28)$$

where  $\mu = (\mu_1, \dots, \mu_{\mathcal{M}+\mathcal{N}})$  is the highest weight of  $\pi_\mu$ . The evaluation morphism associated to the fundamental representation of  $gl(\mathcal{M}|\mathcal{N})$ , with highest weight  $\mu_f = (1, 0, \dots, 0)$ , provides the  $R$  matrix (2.16).

**Theorem 2.3** [22] *Any finite-dimensional irreducible representation of  $\mathcal{Y}(\mathcal{M}|\mathcal{N})$  can be obtained through the tensor products<sup>2</sup> of such evaluation representations.*

Let  $\{ev_{\pi_i}\}_{i=1, \dots, s}$  be a set of evaluation representations. The tensor products of these  $s$  representations  $ev_{\vec{\pi}} = ev_{\pi_1} \otimes \dots \otimes ev_{\pi_s} \circ \Delta^{(s)}$  is a morphism from  $Y(\mathcal{M}|\mathcal{N})$  to the tensor product of  $gl(\mathcal{M}|\mathcal{N})$  representations  $\vec{\pi} = \otimes_i \pi_i$  given by:

$$ev_{\vec{\pi}}(T_{ab}(u)) = \sum_{i_1, \dots, i_{s-1}} ev_{\pi_1}(T_{ai_1}(u)) \otimes ev_{\pi_2}(T_{i_1 i_2}(u)) \otimes \dots \otimes ev_{\pi_s}(T_{i_{s-1} b}(u)) \quad (2.29)$$

### 2.3.2 The generators $T^*(u)$

For the study of superspin chains, we will need also

$$T^*(u) = T^{-1}(u)^t = \sum_{a,b=1}^{\mathcal{M}+\mathcal{N}} T_{ab}^*(u) E_{ab} \quad (2.30)$$

where the graded transposition is defined as

$$A^t = \sum_{i,j=1}^{\mathcal{M}+\mathcal{N}} (-1)^{[i][j]+[j]} A_{ji} E_{ij} = \sum_{i,j=1}^{\mathcal{M}+\mathcal{N}} (A^t)_{ij} E_{ij}, \quad \text{that is} \quad (A^t)_{ij} = (-1)^{[i][j]+[j]} A_{ji}. \quad (2.31)$$

These generators have been introduced by Nazarov [23], and it is easy to see that they obey the same relations as  $T(u)$ :

$$R_{12}(u-v) T_1^*(u) T_2^*(v) = T_2^*(v) T_1^*(u) R_{12}(u-v). \quad (2.32)$$

Thus, the map

$$\varphi : T(u) \mapsto T^*(u) \quad \text{i.e.} \quad \varphi [T_{ij}(u)] = T_{ij}^*(u) = (-1)^{[i][j]+[j]} T_{ji}^{-1}(u) \quad (2.33)$$

is an algebra isomorphism. The exchange relation between  $T^*(u)$  and  $T(v)$  reads

$$R_{12}^{t_1}(v-u) T_1^*(u) T_2(v) = T_2(v) T_1^*(u) R_{12}^{t_1}(v-u), \quad (2.34)$$

$$R_{12}^{t_2}(v-u + \hbar(\mathcal{M} - \mathcal{N})) T_1(u) T_2^*(v) = T_2^*(v) T_1(u) R_{12}^{t_2}(v-u + \hbar(\mathcal{M} - \mathcal{N})), \quad (2.35)$$

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<sup>2</sup>Note however that one has sometimes to make a quotient to get an irreducible representation from these tensor products.

where the superscript  $t_1$  (resp.  $t_2$ ) denotes the transposition in the auxiliary space 1 (resp. 2). We have used the inversion formula

$$R_{12}^{t_2}(x)^{-1} = \frac{-1}{x(x - \hbar)} R_{12}^{t_2}(\hbar(\mathcal{M} - \mathcal{N}) - x). \quad (2.36)$$

One has also

$$\begin{aligned} [T_{nm}^*(u), T_{kl}(v)] &= \frac{\hbar(-1)^{[k][m]}}{u-v} \left( \delta_{ml}(-1)^{[m]+[k][n]} \sum_{a=1}^{\mathcal{M}+\mathcal{N}} (-1)^{[a][n]+[a]} T_{ka}(v) T_{na}^*(u) \right. \\ &\quad \left. - \delta_{nk}(-1)^{[n]} \sum_{a=1}^{\mathcal{M}+\mathcal{N}} (-1)^{[a][m]} T_{am}^*(u) T_{al}(v) \right). \end{aligned} \quad (2.37)$$

### 2.3.3 Liouville contraction and crossing symmetry

The starting point is the equality

$$R_{12}^{t_2}(0) = \hbar Q_{12} = \hbar P_{12}^{t_2} = \hbar \sum_{i,j=1}^{\mathcal{M}+\mathcal{N}} (-1)^{[j]+[i]+[i][j]} E_{ij} \otimes E_{ij}. \quad (2.38)$$

When  $\mathcal{M} \neq \mathcal{N}$ ,  $Q_{12}$  is (up to normalization) a one-dimensional projector  $Q_{12}^2 = (\mathcal{M} - \mathcal{N})Q_{12}$  of  $\text{End}(\mathbb{C}^{(\mathcal{M}|\mathcal{N})})$ . Remark that it is not symmetric:

$$Q_{21} = P_{12} Q_{12} P_{12} = P_{12}^{t_1} = \sum_{i,j=1}^{\mathcal{M}+\mathcal{N}} (-1)^{[i][j]} E_{ij} \otimes E_{ij} \neq Q_{12} = P_{12}^{t_2}. \quad (2.39)$$

Then, from (2.35), one proves that there exist a central element  $Z(u)$  of  $\mathcal{Y}(\mathcal{M}|\mathcal{N})$  such that:

$$Q_{12} T_1(u + \hbar(\mathcal{M} - \mathcal{N})) T_2^*(u) = T_2^*(u) T_1(u + \hbar(\mathcal{M} - \mathcal{N})) Q_{12} = Z(u) Q_{12}. \quad (2.40)$$

We refer to the original work [23] for more details.

Remark that this relation induces a crossing relation for the super-Yangian generators. Indeed, starting from (2.40), one gets

$$Q_{12} T_1(u + \hbar(\mathcal{M} - \mathcal{N})) = Z(u) Q_{12} T_2^*(u)^{-1} \quad (2.41)$$

which, upon transposition in space 2 and multiplication by  $P_{12}$ , leads to

$$\left( (T^{-1}(u)^t)^{-1} \right)^t = \frac{1}{Z(u)} T(u + \hbar(\mathcal{M} - \mathcal{N})), \quad (2.42)$$

or analogously

$$T^t(u)^{-1} = \frac{1}{Z(u - \hbar(\mathcal{M} - \mathcal{N}))} T^{-1}(u - \hbar(\mathcal{M} - \mathcal{N}))^t. \quad (2.43)$$

This relation is nothing but the crossing symmetry for the  $R$ -matrix, but extended at the super-Yangian (abstract) level. It allows a crossing relation for the transfer matrix (see below).



Note that this calculation is also valid for the ‘usual’ Yangian  $\mathcal{Y}(\mathcal{N})$ . In particular, for the  $\mathcal{Y}(\mathcal{M})$  and  $\mathcal{Y}_{-\hbar}(\mathcal{N})$  subalgebras of  $\mathcal{Y}(\mathcal{M}|\mathcal{N})$  one has

$$\left(T_{-\hbar}^{(\mathcal{N})}(u)^t\right)^{-1} = z_{-\hbar}^{(\mathcal{N})}(u) \left(T_{-\hbar}^{(\mathcal{N})}(u + \hbar\mathcal{N})^{-1}\right)^t, \quad (2.44)$$

$$\left(T^{(\mathcal{M})}(u)^t\right)^{-1} = z^{(\mathcal{M})}(u) \left(T^{(\mathcal{M})}(u - \hbar\mathcal{M})^{-1}\right)^t \quad (2.45)$$

for some scalar functions  $z^{(\mathcal{M})}(u)$  and  $z_{-\hbar}^{(\mathcal{N})}(u)$ . They are related to the quantum determinant of  $\mathcal{Y}(\mathcal{M})$  (see e.g. [24]) through:

$$z^{(\mathcal{M})}(u) = \frac{\text{qdet } T^{(\mathcal{M})}(u - \hbar)}{\text{qdet } T^{(\mathcal{M})}(u)}. \quad (2.46)$$

We remind that the quantum determinant  $\text{qdet } T(u)$  is the central element of  $\mathcal{Y}(\mathcal{M})$  given by

$$\text{qdet } T(u) = \sum_{\sigma \in S_{\mathcal{M}}} \text{sgn}(\sigma) T_{\sigma(1)1}(u) \cdots T_{\sigma(\mathcal{M})\mathcal{M}}(u - \hbar(\mathcal{M} - 1)) \quad (2.47)$$

$$= \sum_{\sigma \in S_{\mathcal{M}}} \text{sgn}(\sigma) T_{1\sigma(1)}(u - \hbar(\mathcal{M} - 1)) \cdots T_{\mathcal{M}\sigma(\mathcal{M})}(u). \quad (2.48)$$

Its value in the highest weight representation is computed through application of the above formula on  $v^+$ . For the Yangians  $\mathcal{Y}(\mathcal{M})$  and  $\mathcal{Y}_{-\hbar}(\mathcal{N})$  in  $\mathcal{Y}(\mathcal{M}|\mathcal{N})$ , we get:

$$\text{qdet } T^{(\mathcal{M})}(u) = \lambda_1(u - \hbar(\mathcal{M} - 1)) \cdots \lambda_{\mathcal{M}}(u), \quad (2.49)$$

$$\text{qdet } T_{-\hbar}^{(\mathcal{N})}(u) = \lambda_1^{(\mathcal{N})}(u + \hbar(\mathcal{N} - 1)) \cdots \lambda_{\mathcal{N}}^{(\mathcal{N})}(u), \quad (2.50)$$

where  $\lambda_j^{(\mathcal{N})}(u) = \lambda_{\mathcal{M}+j}(u)$ ,  $j = 1, \dots, \mathcal{N}$ . It leads to the following expressions:

$$z^{(\mathcal{M})}(u) = \frac{\lambda_1(u - \hbar\mathcal{M}) \cdots \lambda_{\mathcal{M}}(u - \hbar)}{\lambda_1(u - \hbar(\mathcal{M} - 1)) \cdots \lambda_{\mathcal{M}}(u)}, \quad (2.51)$$

$$z_{-\hbar}^{(\mathcal{N})}(u) = \frac{\text{qdet } T_{-\hbar}^{(\mathcal{N})}(u + \hbar)}{\text{qdet } T_{-\hbar}^{(\mathcal{N})}(u)} = \frac{\lambda_1^{(\mathcal{N})}(u + \hbar\mathcal{N}) \cdots \lambda_{\mathcal{N}}^{(\mathcal{N})}(u + \hbar)}{\lambda_1^{(\mathcal{N})}(u + \hbar(\mathcal{N} - 1)) \cdots \lambda_{\mathcal{N}}^{(\mathcal{N})}(u)}. \quad (2.52)$$

The calculation of the function  $Z(u)$  needs the use of the quantum Berezinian, see section 2.3.5.

### 2.3.4 Relations for $T^{-1}(u)$

We will need the commutation relations for the inverse of  $T(u)$ , defined by the relation

$$T(u) T^{-1}(u) = \mathbb{I} \quad \text{with} \quad T^{-1}(u) = \sum_{a,b=1}^{\mathcal{M}+\mathcal{N}} T'_{ab}(u) E_{ab}, \quad T'_{ab}(u) = \delta_{ab} + \sum_{n>0} \left(\frac{\hbar}{u}\right)^n T'_{ab}^{(n)}. \quad (2.53)$$

This relation is understood as a series in  $u^{-1}$ , so that expanding the above equality, one can reconstruct the generators  $T'_{ab}^{(n)}$  from the generators  $T_{ab}^{(n)}$ , according to

$$T'_{ab}^{(n)} = -T_{ab}^{(n)} - \sum_{c=1}^{\mathcal{M}+\mathcal{N}} \sum_{p=1}^{n-1} T'_{ac}^{(n-p)} T_{cb}^{(p)}. \quad (2.54)$$

From the relation (2.15), one deduces that

$$T_2(v) R_{12}(v-u) T_1^{-1}(u) = T_1^{-1}(u) R_{12}(v-u) T_2(v), \quad (2.55)$$

$$T_2^{-1}(v) R_{12}(v-u) T_1(u) = T_1(u) R_{12}(v-u) T_2^{-1}(v), \quad (2.56)$$

$$R_{12}(v-u) T_1^{-1}(u) T_2^{-1}(v) = T_2^{-1}(v) T_1^{-1}(u) R_{12}(v-u), \quad (2.57)$$

which upon projection on  $E_{mn} \otimes E_{kl}$  leads to

$$\left[ T'_{mn}(u), T_{kl}(v) \right] = \frac{\hbar (-1)^{[k][n]}}{u-v} \sum_{a=1}^{\mathcal{M}+\mathcal{N}} \left( \delta_{ml} (-1)^{[k][m]+[m][n]} T_{ka}(v) T'_{an}(u) - \delta_{nk} T'_{ma}(u) T_{al}(v) \right).$$

Expanding in  $u^{-1}$  and  $v^{-1}$ , one gets

$$\left[ T'^{(p+1)}_{mn}, T^{(s)}_{kl} \right] = (-1)^{[k][n]} \sum_{r=0}^p \sum_{a=1}^{\mathcal{M}+\mathcal{N}} \left( \delta_{ml} (-1)^{[k][m]+[m][n]} T'^{(s+r)}_{ka} T'^{(p-r)}_{an} - \delta_{nk} T'^{(p-r)}_{ma} T^{(s+r)}_{al} \right). \quad (2.58)$$

**Proposition 2.4** *Let  $v^+$  be a highest weight vector of the super-Yangian. Then,  $v^+$  is also a highest weight vector for  $T^{-1}(u)$ :*

$$T'^{(n)}_{kl} v^+ = 0 \quad \text{for } k > l, \quad 0 < n \quad \text{i.e.} \quad T'_{kl}(u) v^+ = 0 \quad \text{for } k > l, \quad (2.59)$$

$$T'^{(n)}_{kk} v^+ = \lambda'_k{}^{(n)} v^+ \quad \text{for } 0 < n \quad \text{i.e.} \quad T'_{kk}(u) v^+ = \lambda'_k(u) v^+. \quad (2.60)$$

Proof: We make a recursion on  $n$ . Applying (2.54) for  $n = 1$  on  $v^+$ , it is easy to see that (2.59) and (2.60) are true for  $n = 1$ .

Suppose now that we have for a given  $s > 0$  and some scalars  $\lambda'_k{}^{(n)}$

$$\begin{aligned} T'^{(n)}_{kl} v^+ &= 0 \quad \text{for } k > l, \quad 0 < n < s \\ T'^{(n)}_{kk} v^+ &= \lambda'_k{}^{(n)} v^+ \quad \text{for } 0 < n < s, \end{aligned} \quad (2.61)$$

Applying (2.54) for  $n = s$  and  $k > l$  on  $v^+$ , one gets

$$\begin{aligned} T'^{(s)}_{kl} v^+ &= - \sum_{c=1}^l \sum_{p=1}^{s-1} T'^{(s-p)}_{kc} T^{(p)}_{cl} v^+ = - \sum_{c=1}^l \sum_{p=1}^{s-1} \left[ T'^{(s-p)}_{kc}, T^{(p)}_{cl} \right] v^+ = \\ &= \sum_{a=1}^l (-1)^{[a]} \sum_{p=1}^{s-2} p \sum_{c=1}^l \left[ T'^{(s-p-1)}_{kc}, T^{(p)}_{cl} \right] v^+, \end{aligned} \quad (2.62)$$

where to get the last equality, we have used (2.58). Iterating  $r$  times (with  $2 \leq r \leq s-1$ ) this calculation we are led to :

$$T'^{(s)}_{kl} v^+ = A_{l,r} \sum_{p=1}^{s-r-1} B_{s,r,p} \sum_{c=1}^l \left[ T'^{(s-p-r)}_{kc}, T^{(p)}_{cl} \right] v^+.$$

where  $A_{l,r}$  and  $B_{s,r,p}$  are some resummation numbers. Taking  $r = s - 1$  gives (2.59) for  $n = s$ , which is thus proven for all  $n$ .

Finally, applying (2.54) for  $n = s$  and  $k = l$  on  $v^+$ , we have:

$$\begin{aligned} T_{kk}'^{(s)} v^+ &= -\lambda_k^{(s)} v^+ - \sum_{p=1}^{s-1} \lambda_k'^{(s-p)} \lambda_k^{(p)} v^+ + \\ &+ \sum_{c=1}^{k-1} (-1)^{[c]} \sum_{p=1}^{s-1} p \left( \lambda_k'^{(s-p-1)} \lambda_k^{(p)} - \lambda_c'^{(s-p-1)} \lambda_c^{(p)} \right) v^+ + \\ &+ \sum_{c=1}^{k-1} (-1)^{[c]} \sum_{p=1}^{s-2} p \left( \sum_{a=1}^{k-1} \left[ T_{ka}'^{(s-p-1)}, T_{ak}^{(p)} \right] - \sum_{a=1}^{k-1} \left[ T_{ca}^{(p)}, T_{ac}'^{(s-p-1)} \right] \right) v^+. \end{aligned}$$

Again, iterating as in eq. (2.62), we see that only scalar terms acting on  $v^+$  will survive in the r.h.s. This proves the property.  $\blacksquare$

It remains to determine the expression of the eigenvalues  $\lambda'_k(u)$ . This is done in the following proposition:

**Proposition 2.5** *Let  $\lambda'_k(u)$  be the eigenvalue of  $T_{kk}^{-1}(u)$  on  $v^+$ ,  $k = 1, \dots, \mathcal{M} + \mathcal{N}$ . We have*

$$\lambda'_k(u) = \begin{cases} \frac{\lambda_1(u+\hbar) \cdots \lambda_{k-1}(u+\hbar(k-1))}{\lambda_1(u) \cdots \lambda_k(u+\hbar(k-1))}, & k = 1, \dots, \mathcal{M}, \\ Z(u) \frac{\lambda_{k+1}(u+\hbar(2\mathcal{M}-k)) \cdots \lambda_{\mathcal{M}+\mathcal{N}}(u+\hbar(\mathcal{M}-\mathcal{N}+1))}{\lambda_k(u+\hbar(2\mathcal{M}-k)) \cdots \lambda_{\mathcal{M}+\mathcal{N}}(u+\hbar(\mathcal{M}-\mathcal{N}))}, & k = \mathcal{M} + 1, \dots, \mathcal{M} + \mathcal{N}. \end{cases} \quad (2.63)$$

Proof: In order to find the first  $\mathcal{M}$  diagonal entries of  $T^{-1}(u)$ , we start writing

$$\sum_{j \leq k} T_{ij}(u) T_{jk}^{-1}(u) v^+ = \delta_{ik} v^+,$$

and taking  $i, k \leq \mathcal{M}$  we can write, in the distinguished grade,

$$\sum_{j \leq k} (T^{(\mathcal{M})}(u))_{ij} T_{jk}^{-1}(u) v^+ = \delta_{ik} v^+ \quad i, k \leq \mathcal{M}.$$

Considering this as an identity in  $\mathcal{Y}(\mathcal{M}|\mathcal{N})[u^{-1}] \otimes \text{End}(\mathbb{C}^{\mathcal{M}})$ , we can act on the left with  $(T^{(\mathcal{M})}(u))^{-1}$ , obtaining

$$T_{kj}^{-1}(u) v^+ = (T^{(\mathcal{M})}(u))_{kj}^{-1} v^+, \quad k, j = 1, \dots, \mathcal{M}. \quad (2.64)$$

Let us stress that in (2.64),  $T_{kj}^{-1}(u)$  is the entry  $(k, j)$  of the inverse of the  $(\mathcal{M} + \mathcal{N}) \times (\mathcal{M} + \mathcal{N})$  matrix  $T(u)$ , while  $(T^{(\mathcal{M})}(u))_{kj}^{-1}$  is the entry  $(k, j)$  of the inverse of the  $\mathcal{M} \times \mathcal{M}$  matrix  $T^{(\mathcal{M})}(u)$ . In particular, we get the relation

$$T_{kk}^{-1}(u) v^+ = \lambda_k'^{(\mathcal{M})}(u) v^+, \quad k = 1, \dots, \mathcal{M}$$

where the  $\lambda_k^{(\mathcal{M})}(u)$  are the eigenvalues on  $v^+$  of  $(T^{(\mathcal{M})}(u))_{kk}^{-1}$ . It has been shown in [7, 25] that these eigenvalues can be written as

$$\lambda_k^{(\mathcal{M})}(u) = \frac{\lambda_1^{(\mathcal{M})}(u + \hbar) \cdots \lambda_{k-1}^{(\mathcal{M})}(u + \hbar(k-1))}{\lambda_1^{(\mathcal{M})}(u) \cdots \lambda_k^{(\mathcal{M})}(u + \hbar(k-1))}, \quad (2.65)$$

which leads to the first line of eq. (2.63).

For the last  $\mathcal{N}$  diagonal entries of  $T^{-1}(u)$  we start writing in block form the relation  $T^t(u) (T^t(u))^{-1} v^+ = v^+$ , setting

$$T^t(u) = \begin{pmatrix} (T^{(\mathcal{M})}(u))^t & F(u) \\ G(u) & (T_{-\hbar}^{(\mathcal{N})}(u))^t \end{pmatrix}, \quad (T^t(u))^{-1} v^+ = \begin{pmatrix} A(u) & 0 \\ * & D(u) \end{pmatrix} v^+.$$

We then read from the lower right block

$$D(u) v^+ = \left( (T_{-\hbar}^{(\mathcal{N})}(u))^t \right)^{-1} v^+. \quad (2.66)$$

The l.h.s. of this equation is computed via eq. (2.43) which implies, for  $k > \mathcal{M}$ ,

$$(D(u))_{k-\mathcal{M}, k-\mathcal{M}} v^+ = (T^t(u))_{kk}^{-1} v^+ = \frac{1}{Z(u - \hbar(\mathcal{M} - \mathcal{N}))} T'_{kk}(u - \hbar(\mathcal{M} - \mathcal{N})) v^+.$$

The r.h.s. of the equation is computed via eq. (2.44). Comparing the left and right hand sides leads to

$$\lambda'_k(u) = z_{-\hbar}^{(\mathcal{N})}(u + \hbar(\mathcal{M} - \mathcal{N})) Z(u) \lambda'_{k-\mathcal{M}}(u + \hbar\mathcal{M}) \quad k = \mathcal{M} + 1, \dots, \mathcal{M} + \mathcal{N}, \quad (2.67)$$

where the  $\lambda_k^{(\mathcal{N})}(u)$  are the eigenvalues on  $v^+$  of diagonal elements of the  $T_{-\hbar}^{(\mathcal{N})}(u)$  matrix. Applying eq. (2.65) to the  $\mathcal{Y}_{-\hbar}(\mathcal{N})$  subalgebra, we can write these eigenvalues as

$$\lambda_l^{(\mathcal{N})}(u) = \frac{\lambda_1^{(\mathcal{N})}(u - \hbar) \cdots \lambda_{l-1}^{(\mathcal{N})}(u - \hbar(l-1))}{\lambda_1^{(\mathcal{N})}(u) \cdots \lambda_l^{(\mathcal{N})}(u - \hbar(l-1))}, \quad l = 1, \dots, \mathcal{N}.$$

Inserting the value (2.52) of  $z_{-\hbar}^{(\mathcal{N})}$  in eq. (2.67) we find the second line of eq. (2.63). ■

In a finite dimensional irreducible representation, where relations (2.25) hold, we can rewrite eq. (2.63) in the following form:

$$\lambda'_k(u) = \begin{cases} \frac{1}{\lambda_1(u)} \prod_{m=1}^{k-1} \frac{P_m(u + \hbar(m+1))}{P_m(u + \hbar m)}, & k = 1, \dots, \mathcal{M}, \\ \frac{Z(u)}{\lambda_{\mathcal{M}+\mathcal{N}}(u + \hbar(\mathcal{M} - \mathcal{N}))} \prod_{m=k}^{\mathcal{M}+\mathcal{N}-1} \frac{P_m(u + \hbar(2\mathcal{M}-m))}{P_m(u + \hbar(2\mathcal{M}-m+1))}, & k = \mathcal{M} + 1, \dots, \mathcal{M} + \mathcal{N}. \end{cases}$$

### 2.3.5 Quantum Berezinian

The quantum Berezinian was defined by Nazarov [23]. It plays a similar role in the study of the Yangian  $\mathcal{Y}(\mathcal{M}|\mathcal{N})$  as the quantum determinant does in the case of the Yangian  $\mathcal{Y}(\mathcal{N})$ .

**Definition 2.6** *The quantum Berezinian is the following power series with coefficients in the Yangian  $\mathcal{Y}(\mathcal{M}|\mathcal{N})$ :*

$$\begin{aligned} \text{Ber}(u) &= \sum_{\sigma \in S_{\mathcal{M}}} \text{sgn}(\sigma) T_{\sigma(1)1}(u + \hbar(\mathcal{M} - \mathcal{N} - 1)) \cdots T_{\sigma(\mathcal{M})\mathcal{M}}(u - \hbar\mathcal{N}) \\ &\quad \times \sum_{\tau \in S_{\mathcal{N}}} \text{sgn}(\tau) T_{\mathcal{M}+\tau(1),\mathcal{M}+1}^*(u - \hbar\mathcal{N}) \cdots T_{\mathcal{M}+\tau(\mathcal{N}),\mathcal{M}+\mathcal{N}}^*(u - \hbar). \end{aligned} \quad (2.68)$$

One can immediately recognize that

$$\text{Ber}(u) = \text{qdet } T^{(\mathcal{M})}(u + \hbar(\mathcal{M} - \mathcal{N} - 1)) \text{qdet } T^{*(\mathcal{N})}(u - \hbar\mathcal{N}). \quad (2.69)$$

**Proposition 2.7** [23] *The coefficients of the quantum Berezinian (2.68) are central in  $\mathcal{Y}(\mathcal{M}|\mathcal{N})$ . They are related to the Liouville contraction through the identity*

$$\text{Ber}(u) Z(u) = \text{Ber}(u + \hbar). \quad (2.70)$$

The quantum Berezinian being central, one computes its value in the highest weight module by applying expression (2.68) to the h.w. vector  $v^+$ . We get

$$\text{Ber}(u) = \prod_{l=1}^{\mathcal{M}} \lambda_l(u - \hbar\mathcal{N} + \hbar(l-1)) \prod_{l=\mathcal{M}+1}^{\mathcal{M}+\mathcal{N}} \lambda'_l(u - \hbar(\mathcal{M} + \mathcal{N} - l + 1)), \quad (2.71)$$

where the  $\lambda'_l(u)$ ,  $l = \mathcal{M} + 1, \dots, \mathcal{M} + \mathcal{N}$  are given in eq. (2.63). Substitution of this expression in the identity (2.70) yields the following expression for  $Z(u)$ :

$$Z(u) = \frac{\text{Ber}(u + \hbar)}{\text{Ber}(u)} = \prod_{k=1}^{\mathcal{M}} \frac{\lambda_k(u + \hbar k)}{\lambda_k(u + \hbar(k-1))} \prod_{l=\mathcal{M}+1}^{\mathcal{M}+\mathcal{N}} \frac{\lambda_l(u + \hbar(2\mathcal{M} - l))}{\lambda_l(u + \hbar(2\mathcal{M} - l + 1))}. \quad (2.72)$$

Inserting now this expression into eq. (2.63), one obtains:

**Corollary 2.8** *The eigenvalues of the diagonal elements of  $T^{-1}(u)$  on  $v^+$  are given by*

$$\lambda'_k(u) = \frac{\prod_{m=1}^{k-1} \lambda_m(u + \hbar c_m)}{\prod_{m=1}^k \lambda_m(u + \hbar c_{m-1})}, \quad k = 1, \dots, \mathcal{M} + \mathcal{N}. \quad (2.73)$$

where we set  $c_m = \sum_{l=1}^m (-1)^{[l]}$ ,  $m = 1, \dots, \mathcal{M} + \mathcal{N}$ , and  $c_0 = 0$ .

Using expressions (2.71) and (2.73), one gets the value of the quantum Berezinian:

$$\text{Ber}(u) = \prod_{k=1}^{\mathcal{M}} \lambda_k(u + \hbar(k-1)) \prod_{l=\mathcal{M}+1}^{\mathcal{M}+\mathcal{N}} \frac{1}{\lambda_l(u + \hbar(2\mathcal{M} - l))}. \quad (2.74)$$

In what follows, we will also use a different expression for  $Ber(u)$ , proved also in [23]:

$$\begin{aligned} Ber^{-1}(u) &= \sum_{\sigma \in S_{\mathcal{M}}} \text{sgn}(\sigma) T_{\sigma(1)1}^*(u + \hbar(\mathcal{M} - 1)) \cdots T_{\sigma(\mathcal{M})\mathcal{M}}^*(u) \times \\ &\times \sum_{\tau \in S_{\mathcal{N}}} \text{sgn}(\tau) T_{\mathcal{M}+\tau(1), \mathcal{M}+1}(u + \hbar(\mathcal{M} - \mathcal{N})) \cdots T_{\mathcal{M}+\tau(\mathcal{N}), \mathcal{M}+\mathcal{N}}(u + \hbar(\mathcal{M} - 1)). \end{aligned} \quad (2.75)$$

Applying to both factors of expression (2.69) for the quantum Berezinian the known identity (holding in  $\mathcal{Y}_{\hbar}(\mathcal{N})$ )

$$\text{qdet } T(u)A_{\mathcal{N}} = T_{\mathcal{N}}(u - \hbar(\mathcal{N} - 1)) \cdots T_1(u)A_{\mathcal{N}}, \quad (2.76)$$

where  $A_{\mathcal{N}}$  is the normalized antisymmetrizer in the tensor space  $End(\mathbb{C}^{\mathcal{N}})^{\otimes \mathcal{N}}$ , we can write

$$Ber(u)A_{\mathcal{M}}A_{\mathcal{N}} = T_{\mathcal{M}}^{(\mathcal{M})}(u - \hbar\mathcal{N}) \cdots T_1^{(\mathcal{M})}(u + \hbar(\mathcal{M} - \mathcal{N} - 1)) T_{\mathcal{M}+\mathcal{N}}^{*(\mathcal{N})}(u + \hbar') \cdots T_{\mathcal{M}+1}^{*(\mathcal{N})}(u + \hbar'\mathcal{N}) A_{\mathcal{M}}A_{\mathcal{N}},$$

where we have set  $\hbar' = -\hbar$  in the second quantum determinant. The  $A_{\mathcal{M}}$  and  $A_{\mathcal{N}}$  antisymmetrizers are both one-dimensional projectors respectively acting on the tensor product of  $\mathcal{M}$  and  $\mathcal{N}$  copies of the auxiliary space, and can be written in terms of the  $R$  matrices defining  $\mathcal{Y}(\mathcal{M})$  and  $\mathcal{Y}_{\hbar'}(\mathcal{N})$ :

$$\begin{aligned} A_{\mathcal{M}} &= (R_{12} \cdots R_{1\mathcal{M}}) \cdots R_{\mathcal{M}-1, \mathcal{M}}, \quad R_{ij} = R_{ij}^{(\mathcal{M})}(u_i - u_j), \quad u_i - u_{i+1} = \hbar, \\ A_{\mathcal{N}} &= (R'_{\mathcal{M}+1, \mathcal{M}+2} \cdots R'_{\mathcal{M}+1, \mathcal{M}+\mathcal{N}}) \cdots R'_{\mathcal{M}+\mathcal{N}-1, \mathcal{M}+\mathcal{N}}, \quad R'_{ij} = R_{ij}^{(\mathcal{N}), \hbar'}(u'_i - u'_j), \quad u'_i - u'_{i+1} = \hbar'. \end{aligned}$$

Writing now  $T^{(\mathcal{M})}(u) = \mathbb{I}^{(\mathcal{M})}T(u)\mathbb{I}^{(\mathcal{M})}$  and  $T^{*(\mathcal{N})}(u) = \mathbb{I}^{(\mathcal{N})}T^*(u)\mathbb{I}^{(\mathcal{N})}$ , and setting  $\Pi_{\mathcal{M}|\mathcal{N}} = (\mathbb{I}^{(\mathcal{M})})^{\otimes \mathcal{M}} \otimes (\mathbb{I}^{(\mathcal{N})})^{\otimes \mathcal{N}}$ , we get

$$\begin{aligned} Ber(u)A_{\mathcal{M}}A_{\mathcal{N}} &= \Pi_{\mathcal{M}|\mathcal{N}} T_{\mathcal{M}}(u - \hbar\mathcal{N}) \cdots T_1(u + \hbar(\mathcal{M} - \mathcal{N} - 1)) \times \\ &\times T_{\mathcal{M}+\mathcal{N}}^*(u - \hbar) \cdots T_{\mathcal{M}+1}^*(u - \hbar\mathcal{N}) A_{\mathcal{M}}A_{\mathcal{N}}. \end{aligned}$$

The same steps applied to eq. (2.75) lead to the following equation.

$$\begin{aligned} Ber^{-1}(u)A_{\mathcal{M}}A_{\mathcal{N}} &= \Pi_{\mathcal{M}|\mathcal{N}} T'_{\mathcal{M}}(u + \hbar(\mathcal{M} - 1)) \cdots T'_1(u) \times \\ &\times T_{\mathcal{M}+\mathcal{N}}(u + \hbar(\mathcal{M} - 1)) \cdots T_{\mathcal{M}+1}(u + \hbar(\mathcal{M} - \mathcal{N})) A_{\mathcal{M}}A_{\mathcal{N}}. \end{aligned}$$

The above expressions can be considered as the graded counterparts of eq. (2.76): both relations act on a number of copies of the auxiliary space equal to the dimension of the Yangian and relate a  $(\mathcal{M} + \mathcal{N})$ -fold tensor product of  $T$  matrices to a central element by means of suitable one-dimensional projectors.

## 2.4 Reflection superalgebra

To study (soliton-preserving) open spin chains, we need to introduce another algebraic structure, the reflection algebra. It is a subalgebra of the super-Yangian, and actually can be defined from any quantum group. Focusing on the super-Yangian, the reflection superalgebra is a subalgebra of  $\mathcal{Y}(\mathcal{M}|\mathcal{N})$ , built as follows. One starts to consider

$$B(u) = T(u) K(u) T^{-1}(-u) \quad (2.77)$$

where  $T(u)$  generates the super-Yangian and  $K(u)$  is a matrix obeying the graded reflection (boundary Yang–Baxter) equation

$$R_{12}(u_1 - u_2) K_1(u_1) R_{21}(u_1 + u_2) K_2(u_2) = K_2(u_2) R_{12}(u_1 + u_2) K_1(u_1) R_{21}(u_1 - u_2). \quad (2.78)$$

Using the exchange relations (2.15), it is easy to deduce that  $B(u)$  also obeys the graded reflection equation

$$R_{12}(u_1 - u_2) B_1(u_1) R_{21}(u_1 + u_2) B_2(u_2) = B_2(u_2) R_{12}(u_1 + u_2) B_1(u_1) R_{21}(u_1 - u_2), \quad (2.79)$$

or, in components:

$$\begin{aligned} [B_{ij}(u), B_{kl}(v)] &= \frac{(-1)^{\eta(i,j,k)} \hbar}{u - v} (B_{kj}(u) B_{il}(v) - B_{kj}(v) B_{il}(u)) \\ &+ \frac{\hbar}{u + v} \left( (-1)^{[j]} \delta_{jk} \sum_{a=1}^{\mathcal{M}+\mathcal{N}} B_{ia}(u) B_{al}(v) - (-1)^{\eta(i,j,k)} \delta_{il} \sum_{a=1}^{\mathcal{M}+\mathcal{N}} B_{ka}(v) B_{aj}(u) \right) \\ &- \frac{\hbar^2}{u^2 - v^2} \delta_{ij} \left( \sum_{a=1}^{\mathcal{M}+\mathcal{N}} B_{ka}(u) B_{al}(v) - \sum_{a=1}^{\mathcal{M}+\mathcal{N}} B_{ka}(v) B_{al}(u) \right). \end{aligned} \quad (2.80)$$

This relation shows that  $B(u)$  generates a subalgebra of the super-Yangian, called reflection algebra and denoted  $\mathfrak{B}$ .

Using the coproduct (2.22), one then shows that

$$\Delta(B_{ij}(u)) = \sum_{l,m=1}^{\mathcal{M}+\mathcal{N}} (-1)^{([m]+[j])([m]+[l])} T_{il}(u) T'_{mj}(-u) \otimes B_{lm}(u). \quad (2.81)$$

This proves that the reflection algebra is a Hopf coideal of  $\mathcal{Y}(\mathcal{M}|\mathcal{N})$ :

$$\Delta(\mathfrak{B}) \subseteq \mathcal{Y}(\mathcal{M}|\mathcal{N}) \otimes \mathfrak{B}.$$

This will allow us to define monodromy matrices for open spin chains (see section 5.1 below). In this context, the matrix  $K(u)$  will be related to the boundary condition of the spin chain. Hence, the classification of  $K$  matrices is essential in the study of open spin chains. As far as the super-Yangian is concerned, they have been classified in [26]. The result is summarized in the following proposition

**Proposition 2.9** *Any invertible solution of the soliton preserving reflection equation (2.78) takes the form  $K(u) = U \left( \mathbb{E} + \frac{\xi}{u} \mathbb{I} \right) U^{-1}$  where either*

1.  $\mathbb{E}$  is diagonal and  $\mathbb{E}^2 = \mathbb{I}$  (diagonalizable solutions)
2.  $\mathbb{E}$  is strictly triangular and  $\mathbb{E}^2 = 0$  (non-diagonalizable solutions)

The matrix  $U$  is an element of the group  $GL(\mathcal{M}) \times GL(\mathcal{N})$ , independent of the spectral parameter;  $\xi$  is a free parameter, and the classification is done up to multiplication by a function of the spectral parameter.

We will restrict to the case of diagonalizable solutions. The possible matrices  $\mathbb{E}$  are then labeled by two integers  $L_1$  and  $L_2$ ,  $0 \leq L_1 \leq L_2 \leq \mathcal{M} + \mathcal{N}$ , which count the number of  $-1$  on the diagonal of  $\mathbb{E}$ :

$$\mathbb{E} = \text{diag} \left( \underbrace{-1, \dots, -1}_{L_1}, \underbrace{1, \dots, 1}_{L_2 - L_1}, \underbrace{-1, \dots, -1}_{\mathcal{N} + \mathcal{M} - L_2} \right) \equiv \text{diag} (\theta_1, \dots, \theta_{\mathcal{M} + \mathcal{N}}).$$

Let us stress that the diagonalization matrix  $U$  being constant, it is sufficient to consider diagonal  $K(u)$  matrices: the other cases are recovered by a conjugation  $T(u) \rightarrow U^{-1} T(u) U$  on each site of the chain, which does not affect the reflection algebra, nor the transfer matrix [26]. The algebraic structure of  $\mathfrak{B}$  does depend on the choice for  $K(u)$ . Indeed, from the expansion

$$B_{ij}(u) = \theta_i \delta_{ij} + \frac{1}{u} \left( (\theta_i + \theta_j) T_{ij}^{(1)} - \xi \delta_{ij} \right) + \frac{1}{u^2}(\dots). \quad (2.82)$$

we deduce that, when  $L_1 \leq \mathcal{M} \leq L_2$ , the Lie sub-superalgebra in  $\mathfrak{B}$  is  $gl(L_1 | \mathcal{M} + \mathcal{N} - L_2) \oplus gl(\mathcal{M} - L_1 | L_2 - \mathcal{M})$ . Hence, the notation  $\mathfrak{B}$  should also contain the labels  $\mathcal{N}, \mathcal{M}, L_1, L_2$ : we omit them for simplicity.

In the following, we will choose the normalisation of the resulting reflection matrix in such a way that its entries are analytical:

$$K(u) = \text{diag} \left( \underbrace{\xi - u, \dots, \xi - u}_{L_1 \text{ terms}}, \underbrace{u + \xi, \dots, u + \xi}_{L_2 - L_1 \text{ terms}}, \xi - u, \dots, \xi - u \right). \quad (2.83)$$

#### 2.4.1 Highest weight representations of the reflection algebra

We construct highest weight representations of the reflection superalgebras based on those of the super-Yangian. This construction will be used later on to build open spin chains. However, a complete classification, similar to the one done in [25] for reflection algebras (based on the Yangian  $\mathcal{Y}(\mathcal{N})$ ), remains to be done.

**Proposition 2.10** *The vector  $v^+$  is a highest weight vector for the representations of the reflection algebra obtained from the representation (2.24) of  $\mathcal{Y}(\mathcal{M} | \mathcal{N})$  with:*

$$B_{kl}(u) v^+ = 0, \quad 1 \leq l < k \leq \mathcal{M} + \mathcal{N}, \quad (2.84)$$

$$B_{kk}(u) v^+ = \frac{2u}{2u - \hbar c_{k-1}} g_k(u) \lambda_k(u) \lambda'_k(-u) v^+ - \sum_{j=1}^{k-1} g_j(u) a_j(u) v^+, \quad 1 \leq k \leq \mathcal{M} + \mathcal{N}, \quad (2.85)$$

where  $c_k = \sum_{a=1}^k (-1)^{[a]}$  and

$$g_k(u) = \begin{cases} (\xi - u), & \text{if } 1 \leq k \leq L_1 \\ (\xi + u - \hbar c_{L_1}), & \text{if } L_1 < k \leq L_2, \\ (\xi - u - \hbar(c_{L_1} - c_{L_2})), & \text{if } L_2 < k \leq \mathcal{M} + \mathcal{N}, \end{cases} \quad (2.86)$$

$$a_k(u) = (-1)^{[k]} \hbar \frac{2u \lambda_k(u) \lambda'_k(-u)}{(2u - \hbar c_k)(2u - \hbar c_{k-1})}. \quad (2.87)$$



Proof: We start writing, for  $k > l$ ,

$$B_{kl}(u) v^+ = \sum_{j=1}^l T_{kj}(u) K_{jj}(u) T'_{jl}(-u) v^+ = \sum_{j=1}^l K_{jj}(u) [T_{kj}(u), T'_{jl}(-u)] v^+. \quad (2.88)$$

From the commutation relations, we find for  $a \leq l < k$

$$[T_{ka}(u), T'_{al}(-u)] v^+ = (-1)^{[a]} \frac{\hbar}{2u} \sum_{b=1}^l T_{kb}(u) T'_{bl}(-u) v^+, \quad (2.89)$$

Considering the case  $a = l$ , we see that the l.h.s. of (2.89) vanishes, so that

$$\sum_{b=1}^l T_{kb}(u) T'_{bl}(-u) v^+ = 0.$$

Hence the right hand side of eq. (2.88) also vanishes, proving (2.84).

We now turn to the case  $l = k$ , i.e. to the eigenvalues of  $B_{kk}(u)$  on  $v^+$ . We start defining

$$f_a(u) \doteq \sum_{k=1}^a T'_{ak}(-u) T_{ka}(u) v^+ \quad \text{and} \quad \Psi_i(u) \doteq \sum_{k=1}^i T_{ik}(u) T'_{ki}(-u) v^+.$$

The supercommutation relations applied to these definitions imply

$$\begin{cases} f_a(u) = \frac{1}{2u - \hbar c_{a-1}} \left( 2u \lambda_a(u) \lambda'_a(-u) v^+ - \hbar \sum_{k=1}^{a-1} (-1)^{[k]} \Psi_k(u) \right) \\ \Psi_a(u) = \frac{1}{2u - \hbar c_{a-1}} \left( 2u \lambda_a(u) \lambda'_a(-u) v^+ - \hbar \sum_{k=1}^{a-1} (-1)^{[k]} f_k(u) \right), \end{cases} \quad (2.90)$$

for  $a = 1, \dots, \mathcal{M} + \mathcal{N}$ . Since  $f_1(u) = \Psi_1(u) = \lambda_1(u) \lambda'_1(-u) v^+$ , the system (2.90) has a unique solution  $f_a(u) = \Psi_a(u)$ , so we can rewrite the expression of  $f_a(u)$  as

$$\left( 1 - \frac{\hbar}{2u} c_{a-1} \right) f_a(u) = \lambda_a(u) \lambda'_a(-u) v^+ - \frac{\hbar}{2u} \sum_{k=1}^{a-1} (-1)^{[k]} f_k(u). \quad (2.91)$$

Eq.(2.91) is a triangular linear system in the unknowns  $f_a(u)$  whose unique solution can be written as:

$$f_j(u) = \frac{\lambda_j(u) \lambda'_j(-u)}{1 - \frac{\hbar}{2u} c_{j-1}} v^+ - \sum_{l=1}^{j-1} \frac{(-1)^{[l]} \hbar \lambda_l(u) \lambda'_l(-u)}{2u \left( 1 - \frac{\hbar}{2u} c_l \right) \left( 1 - \frac{\hbar}{2u} c_{l-1} \right)} v^+ = \frac{\lambda_j(u) \lambda'_j(-u)}{1 - \frac{\hbar}{2u} c_{j-1}} v^+ - \sum_{l=1}^{j-1} a_l(u) v^+. \quad (2.92)$$

Using this expression it is now clear that for  $j \leq L_1$  we can write:

$$B_{jj}(u) v^+ = (\xi - u) f_j(u) = \left( \frac{2u(\xi - u) \lambda_j(u) \lambda'_j(-u)}{2u - \hbar c_{j-1}} - (\xi - u) \sum_{k=1}^{j-1} a_k(u) \right) v^+.$$

For  $L_1 < j \leq L_2$  we have

$$\begin{aligned}
B_{jj}(u) v^+ &= (\xi + u) f_j(u) - 2u \sum_{k=1}^{L_1} T_{jk}(u) T'_{kj}(-u) v^+ \\
&= (\xi + u - \hbar c_{L_1}) f_j(u) + \hbar \sum_{k=1}^{L_1} (-1)^{[k]} f_k(u), \tag{2.93}
\end{aligned}$$

where to get the last equality we have used supercommutation relations on  $T_{jk}(u) T'_{kj}(-u)$ . Using now eq. (2.92), we get

$$\hbar \sum_{k=1}^{L_1} (-1)^{[k]} f_k(u) = (2u - \hbar c_{L_1}) \sum_{k=1}^{L_1} a_k(u) v^+.$$

Substituting the above equation in eq. (2.93), we get the required result. An analogous calculation for the  $j > L_2$  case leads to (2.86). ■

### 3 Closed super-spin chains

#### 3.1 Monodromy and transfer matrices

One defines the ( $L$  sites) monodromy matrix  $\mathcal{T}(u)$  as:

$$\mathcal{T}(u) = \Delta^{(L)}(T(u)) = T(u) \otimes T(u) \otimes \cdots \otimes T(u) \in \text{End}(\mathbb{C}^{\mathcal{M}|\mathcal{N}}) \otimes (\mathcal{Y}(\mathcal{M}|\mathcal{N}))^{\otimes L}. \tag{3.1}$$

Applying an evaluation map on each term of this tensor product provides the ‘usual’ monodromy matrix: the different sites correspond to the terms in the tensor product, and the evaluation map defines the ‘spin’ (the representation) carried by the site. Taking different representations of the super-Yangian allows to construct various type of closed super-spin chain models.

From the relation (2.15), it is easy to show that both the trace and the supertrace of the monodromy matrix

$$t(u) = \text{tr}_a \mathcal{T}(u) = \sum_{i=1}^{\mathcal{M}+\mathcal{N}} \mathcal{T}_{ii}(u) \quad \text{and} \quad st(u) = \text{str}_a \mathcal{T}(u) = \sum_{i=1}^{\mathcal{M}+\mathcal{N}} (-1)^{[i]} \mathcal{T}_{ii}(u) \tag{3.2}$$

generate commutative families of operators:

$$[t(u), t(v)] = 0 \quad \text{and} \quad [st(u), st(v)] = 0. \tag{3.3}$$

Note however that  $t(u)$  and  $st(u)$  do not commute one with each other. Hence, they will generate different families of commuting observables.

### 3.2 Global invariance of transfer matrices

Taking the supertrace on the auxiliary space 1 in relation (2.15), one is left with

$$[X, st(u)] = 0, \quad \forall X \in gl(\mathcal{M}|\mathcal{N}). \quad (3.4)$$

On the other hand, taking the trace in (2.15) leads to

$$[T_{kl}^{(1)}, t(u)] = ((-1)^{[i]} - (-1)^{[k]}) T_{kl}(u), \quad (3.5)$$

which is obviously zero iff  $l$  and  $k$  are both even or odd indices:

$$[X, t(u)] = 0, \quad \forall X \in gl(\mathcal{M}) \oplus gl(\mathcal{N}). \quad (3.6)$$

Then, the transfer matrix  $st(u)$  enjoys the full  $gl(\mathcal{M}|\mathcal{N})$  symmetry, while the transfer matrix  $t(u)$  is only  $gl(\mathcal{M}) \oplus gl(\mathcal{N})$  invariant.

It is thus reasonable to think that the models associated to  $st(u)$  are more relevant than the ones associated to  $t(u)$  for the construction of super-spin chain models. We will nevertheless present the Bethe ansatz for both transfer matrices. Note however that the construction of open spin chain models is possible for the supertrace only, emphasising the difference between  $t(u)$  and  $st(u)$ .

### 3.3 Pseudovacuum for transfer matrices

Starting from a  $\mathcal{Y}(\mathcal{M}|\mathcal{N})$  highest weight vector it is possible to construct an eigenvector of the transfer matrix. If  $V_1, \dots, V_L$  are highest weight modules for  $\mathcal{Y}(\mathcal{M}|\mathcal{N})$ , with highest weight vectors  $v_1, \dots, v_L$ , then the vector  $v^+ \doteq v_1 \otimes \dots \otimes v_L$  is a highest weight vector for the monodromy matrix, and thus an eigenvector of the transfer matrices:

$$\mathcal{T}_{ij}(u) v^+ = 0, \quad 1 \leq j < i \leq \mathcal{M} + \mathcal{N}, \quad (3.7)$$

$$\mathcal{T}_{kk}(u) v^+ = \left( \prod_{n=1}^L \lambda_k^{[n]}(u) \right) v^+ \doteq \lambda_k(u) v^+. \quad (3.8)$$

Eq. (3.8) allows to compute the eigenvalue of  $st(u)$ :

$$t(u) v^+ = \widehat{\Lambda}_0(u) v^+, \quad \text{with} \quad \widehat{\Lambda}_0(u) \doteq \sum_{k=1}^{\mathcal{M}+\mathcal{N}} \lambda_k(u) = \sum_{k=1}^{\mathcal{M}+\mathcal{N}} \prod_{n=1}^L \lambda_k^{[n]}(u), \quad (3.9)$$

$$st(u) v^+ = \Lambda_0(u) v^+, \quad \text{where} \quad \Lambda_0(u) \doteq \sum_{k=1}^{\mathcal{M}+\mathcal{N}} (-1)^{[k]} \lambda_k(u) = \sum_{k=1}^{\mathcal{M}+\mathcal{N}} (-1)^{[k]} \prod_{n=1}^L \lambda_k^{[n]}(u). \quad (3.10)$$

Using evaluation representations (2.26),  $ev_{\pi_n}$  for  $1 \leq n \leq L$ , with highest weight

$$\lambda_k^{[i]}(u) = 1 + (-1)^{[k]} \frac{\hbar}{u - a_i} \mu_k^{[i]},$$

we easily get the highest weight of the representation:

$$\begin{aligned} ev_{\pi}(\mathcal{T}_{kk}(u)) v^+ &= \prod_{n=1}^L \left( 1 + (-1)^{[k]} \frac{\hbar}{u - a_n} \mu_k^{[n]} \right) v^+, \quad k = 1, \dots, \mathcal{M} + \mathcal{N}, \\ ev_{\pi}(st(u)) v^+ &= \sum_{k=1}^{\mathcal{M} + \mathcal{N}} (-1)^{[k]} \prod_{n=1}^L \left( 1 + (-1)^{[k]} \frac{\hbar}{u - a_n} \mu_k^{[n]} \right) v^+. \end{aligned}$$

It is important for what follows to remark that the above relations imply that the entries of the matrix  $(u - a_n)T(u)$  in a  $ev_{\pi}$  representation are analytical. From now on, we will use for the local and monodromy matrices the normalizations:

$$T_k^{[n]}(u) \mapsto (u - a_n) T_k^{[n]}(u), \quad \text{and} \quad \mathcal{T}(u) \mapsto \prod_{n=1}^L (u - a_n) \mathcal{T}(u), \quad (3.11)$$

that ensure analyticity of their entries. The transfer matrix will be accordingly normalized. Notice that with the normalization (3.11) the highest weight in the  $ev_{\pi_n}$  representation reads:

$$\lambda_k^{[n]}(u) = u - a_n + (-1)^{[k]} \hbar \mu_k^{[n]} \quad \text{and} \quad \lambda_k(u) = \prod_{n=1}^L \left( u - a_n + (-1)^{[k]} \hbar \mu_k^{[n]} \right). \quad (3.12)$$

Nevertheless, let us stress the fact that the above calculation only relies on the existence of a highest weight vector, and thus remains valid for infinite dimensional (highest weight) representations. When the representations are finite dimensional, it is possible to rewrite  $\Lambda_0(u)$  in terms of Drinfeld polynomials. Indeed, we will see that the BAEs depend on the representation only through the Drinfeld polynomials.

### 3.4 Dressing hypothesis

Having determined the form of the pseudovacuum eigenvalue we assume now the following form for the general transfer matrix eigenvalues:

$$\widehat{\Lambda}(u) = \sum_{k=1}^{\mathcal{M} + \mathcal{N}} \lambda_k(u) \widehat{A}_{k-1}(u), \quad (3.13)$$

$$\Lambda(u) = \sum_{k=1}^{\mathcal{M} + \mathcal{N}} (-1)^{[k]} \lambda_k(u) A_{k-1}(u), \quad (3.14)$$

where the so-called dressing functions  $A_i(u)$  and  $\widehat{A}_i(u)$ ,  $i = 0, \dots, \mathcal{M} + \mathcal{N} - 1$  are to be determined implementing a number of constraints upon the spectrum:

1. the  $R$  matrix and monodromy matrix being written in terms of rational functions of the spectral parameter  $u$ , one assumes that  $A_l(u)$ ,  $\forall l$ , are also rational functions;
2. analyticity requirements imposed on the spectrum lead to the assumption that  $A_l(u)$  (resp.  $\widehat{A}_l(u)$ ) has common poles with  $A_{l\pm 1}(u)$  (resp.  $\widehat{A}_{l\pm 1}(u)$ ) only;

3. the poles of the dressing functions will be assumed simple: the relation between  $A_l(u)$  and  $A_{l+1}(u)$  poles is the simplest one which ensures the analyticity of the eigenvalues;
4. the asymptotic expansion of the transfer matrix will provide information about the number of factors in the aforementioned rational functions;
5. the generalized fusion provides relations among the dressing functions.

Requirements 1. and 2. fix the following form for the dressing functions:

$$A_l(u) = \begin{cases} \prod_{j=1}^{M^{(l)}} \frac{u - \alpha_j^{(l)}}{u - u_j^{(l)} - \hbar \frac{l}{2}} \prod_{j=1}^{M^{(l+1)}} \frac{u - \beta_j^{(l+1)}}{u - u_j^{(l+1)} - \hbar \frac{l+1}{2}}, & 0 \leq l < \mathcal{M}, \\ \prod_{j=1}^{M^{(l)}} \frac{u - \alpha_j^{(l)}}{u - u_j^{(l)} - \hbar (\mathcal{M} - \frac{l}{2})} \prod_{j=1}^{M^{(l+1)}} \frac{u - \beta_j^{(l+1)}}{u - u_j^{(l+1)} - \hbar (\mathcal{M} - \frac{l+1}{2})}, & \mathcal{M} \leq l < \mathcal{M} + \mathcal{N}, \end{cases} \quad (3.15)$$

where  $M^{(0)} = M^{(\mathcal{M}+\mathcal{N})} = 0$ , while the values of the integers  $M^{(l)}$ ,  $l = 1, \dots, \mathcal{M} + \mathcal{N} - 1$  are to be determined by means of asymptotic expansion (point 4. above), as will be shown in the next section; the shifts in the denominators have been introduced for later convenience.

The next step consists in finding constraints to determine  $\alpha_j^{(l)}$  and  $\beta_j^{(l)}$  in terms of  $u_j^{(l)}$ . This is achieved by means of the generalized fusion procedure.

### 3.4.1 Values of the $gl(\mathcal{M}|\mathcal{N})$ Cartan generators

As we have seen in section 3.2, the generators of the finite-dimensional  $gl(\mathcal{M}|\mathcal{N})$  superalgebra commute with the transfer matrix. It is thus possible to relate the integers  $M^{(l)}$ ,  $l = 1, \dots, \mathcal{M} + \mathcal{N} - 1$ , appearing in the  $\Lambda(u)$  dressing to the eigenvalues of the Cartan generators of  $gl(\mathcal{M}|\mathcal{N})$ . This can be done in the following way.

Taking first the  $u \rightarrow \infty$  in the expression (3.14) for  $\Lambda(u)$  for an  $L$  sites chain, one gets

$$\Lambda(u) \sim u^L (\mathcal{M} - \mathcal{N}) + u^{L-1} \sum_{k=1}^{\mathcal{M}+\mathcal{N}} (-1)^{[k]} \hbar \left( \lambda_k^{(1)} - M^{(k-1)} + M^{(k)} \right),$$

where we set  $\lambda_k(u) = u + \hbar \lambda_k^{(1)} + O\left(\frac{1}{u}\right)$ . On the other hand, the same expansion performed on the transfer matrix  $st(u)$  leads to

$$st(u) \sim u^L (\mathcal{M} - \mathcal{N}) + u^{L-1} \sum_{k=1}^{\mathcal{M}+\mathcal{N}} \hbar \left( \sum_{n=1}^L \mathcal{E}_k^{[n]} \right),$$

where  $\sum_{n=1}^L \mathcal{E}_k^{[n]} = \sum_{n=1}^L (-1)^{[k]} T_{kk}^{(1)[n]}$  is the  $k$ -th diagonal generator of the global  $gl(\mathcal{M}|\mathcal{N})$  symmetry algebra of the chain. Starting then from a transfer matrix eigenvector with eigenvalue (3.14), one can write

$$(-1)^{[k]} h_k = \lambda_k^{(1)} - M^{(k-1)} + M^{(k)},$$

where  $h_k$  is the eigenvalue of the diagonal generator  $\sum_{n=1}^L \mathcal{E}_k^{[n]}$ . For the Cartan generators of  $gl(\mathcal{M}|\mathcal{N})$ ,  $s_k = (-1)^{[k]} \mathcal{E}_k - (-1)^{[k+1]} \mathcal{E}_{k+1}$ , one gets

$$s_k v = \left( 2M^{(k)} - M^{(k-1)} - M^{(k+1)} + \lambda_k^{(1)} - \lambda_{k+1}^{(1)} \right) v.$$

The above calculation shows that the values of the  $M^{(k)}$  integers are related to the conserved charges of the global symmetry of the chain: one must then take care that simplifications in the dressing functions resulting from the fusion procedure do not change their number of factors. In other words each  $M^{(k)}$  should be kept independent from each other and only relations between the other parameters appearing in the dressing are allowed, as we will shown in the next section.

### 3.4.2 Generalized fusion from quantum Berezinian

The relations (2.32), (2.34) and (2.35), between  $T^*(u)$  and  $T(v)$  show that we can define another transfer matrix  $st^*(u) = str T^*(u)$  which obeys

$$[st(u), st^*(v)] = 0 \quad \text{and} \quad [st^*(u), st^*(v)] = 0 \quad (3.16)$$

so that one can consider the dressing of  $st^*(u)$  simultaneously with the one of  $st(v)$ :

$$\Lambda^*(u) = \sum_{k=1}^{\mathcal{M}+\mathcal{N}} (-1)^{[k]} \lambda_k^*(u) A_k^*(u), \quad (3.17)$$

where  $T_{kk}^*(u) v^+ = \lambda_k^*(u) v^+$  and

$$A_l^*(u) = \begin{cases} \prod_{j=1}^{M^{(l)}} \frac{u - \alpha_j^{*(l)}}{u - u_j^{*(l)} - \hbar \left(\mathcal{M} - \frac{l}{2}\right)} \prod_{j=1}^{M^{(l+1)}} \frac{u - \beta_j^{*(l+1)}}{u - u_j^{*(l+1)} - \hbar \left(\mathcal{M} - \frac{l+1}{2}\right)}, & 0 \leq l < \mathcal{M}, \\ \prod_{j=1}^{M^{(l)}} \frac{u - \alpha_j^{*(l)}}{u - u_j^{*(l)} - \hbar \frac{l}{2}} \prod_{j=1}^{M^{(l+1)}} \frac{u - \beta_j^{*(l+1)}}{u - u_j^{*(l+1)} - \hbar \frac{l+1}{2}}, & \mathcal{M} \leq l < \mathcal{M} + \mathcal{N}. \end{cases}$$

Let  $A_{\mathcal{M}}, A_{\mathcal{N}}, \Pi_{\mathcal{M}|\mathcal{N}}$  be the one-dimensional projectors defined in section 2.3.5 which act on auxiliary spaces  $1, \dots, \mathcal{M} + \mathcal{N}$  and denote

$$\mathcal{T}\mathcal{T}^* = \mathcal{T}_{\mathcal{M}}(u - \hbar\mathcal{N}) \cdots \mathcal{T}_1(u + \hbar(\mathcal{M} - \mathcal{N} - 1)) \mathcal{T}_{\mathcal{M}+\mathcal{N}}^*(u - \hbar) \cdots \mathcal{T}_{\mathcal{M}+1}^*(u - \hbar\mathcal{N}).$$

Then, from the following relation

$$\mathcal{T}\mathcal{T}^* = Ber(u) A_{\mathcal{M}} A_{\mathcal{N}} + (1 - \Pi_{\mathcal{M}|\mathcal{N}}) \mathcal{T}\mathcal{T}^* A_{\mathcal{M}} A_{\mathcal{N}} + \mathcal{T}\mathcal{T}^* (1 - A_{\mathcal{M}} A_{\mathcal{N}}), \quad (3.18)$$

we deduce, by taking the supertrace in the spaces  $1, \dots, \mathcal{M} + \mathcal{N}$ , that

$$st(u - \hbar\mathcal{N}) \cdots st(u + \hbar(\mathcal{M} - \mathcal{N} - 1)) st^*(u - \hbar) \cdots st^*(u - \hbar\mathcal{N}) = (-1)^{\mathcal{N}} Ber(u) + st_{\mathfrak{f}}^{(1)}(u),$$

where  $st_{\mathfrak{f}}^{(1)}(u) = str_{1 \dots \mathcal{M}+\mathcal{N}} [(1 - \Pi_{\mathcal{M}|\mathcal{N}}) \mathcal{T}\mathcal{T}^* A_{\mathcal{M}} A_{\mathcal{N}} + \mathcal{T}\mathcal{T}^* (1 - A_{\mathcal{M}} A_{\mathcal{N}})]$  is a so-called fused transfer matrix. Then, acting with relation (3.18) on any  $(st(u)$  and  $st^*(u))$  eigenvector  $v$  with eigenvalues  $\Lambda(u)$ ,  $\Lambda^*(u)$ , one obtains

$$\begin{aligned} & \Lambda(u - \hbar\mathcal{N}) \cdots \Lambda(u + \hbar(\mathcal{M} - \mathcal{N} - 1)) \Lambda^*(u - \hbar) \cdots \Lambda^*(u - \hbar\mathcal{N}) = \\ & = (-1)^{\mathcal{N}} \prod_{k=1}^{\mathcal{M}} \lambda_k(u - \hbar(\mathcal{N} - k + 1)) \prod_{l=\mathcal{M}+1}^{\mathcal{M}+\mathcal{N}} \lambda'_l(u + \hbar(\mathcal{M} + \mathcal{N} - l + 1)) + \Lambda_{\mathfrak{f}}^{(1)}(u), \end{aligned} \quad (3.19)$$

where  $\Lambda_{\mathfrak{f}}^{(1)}(u) v = st_{\mathfrak{f}}^{(1)}(u) v$  and we have used eq. (2.71). Let us remark that this relation shows that  $v$  is also an eigenvector of  $t_{\mathfrak{f}}^{(1)}(u)$ . Using the postulated expression (3.14) for the eigenvalues and picking the term proportional to  $\prod_{k=1}^{\mathcal{M}} \lambda_k(u - \hbar(\mathcal{N} - k + 1)) \prod_{l=\mathcal{M}+1}^{\mathcal{M}+\mathcal{N}} \lambda'_l(u + \hbar(\mathcal{M} + \mathcal{N} - l + 1))$  in eq. (3.19), we deduce a first constraint between the dressing functions, namely

$$A_0(u - \hbar\mathcal{N}) \cdots A_{\mathcal{M}-1}(u + \hbar(\mathcal{M} - \mathcal{N} - 1)) A_{\mathcal{M}}^*(u - \hbar\mathcal{N}) \cdots A_{\mathcal{M}+\mathcal{N}-1}^*(u - \hbar) = 1. \quad (3.20)$$

The simplest non-trivial choice of the  $\alpha_j^{(k)}, \alpha_j^{*(k)}$  and  $\beta_j^{(k)}, \beta_j^{*(k)}$  satisfying this constraint is to set  $\alpha_j^{(k)} = u_j^{(k)} + \frac{\hbar}{2}(k+2)$ ,  $\beta_j^{(k+1)} = u_j^{(k+1)} + \frac{\hbar}{2}(k-1)$ ,  $\forall j$ , for  $k = 0, \dots, \mathcal{M}-1$ ,  $u_j^{*(\mathcal{M})} = u_j^{(\mathcal{M})} - \hbar\mathcal{M}$ , and  $\alpha_j^{*(k)} = u_j^{*(k)} + \frac{\hbar}{2}(k+2)$ ,  $\beta_j^{*(k+1)} = u_j^{*(k+1)} + \frac{\hbar}{2}(k-1)$ ,  $\forall j$ , for  $k = \mathcal{M}, \dots, \mathcal{M} + \mathcal{N} - 1$  in such a way that

$$\begin{aligned} A_k(u) &= \prod_{j=1}^{M^{(k)}} \frac{u - u_j^{(k)} - \hbar \frac{k+2}{2}}{u - u_j^{(k)} - \hbar \frac{k}{2}} \prod_{j=1}^{M^{(k+1)}} \frac{u - u_j^{(k+1)} - \hbar \frac{k-1}{2}}{u - u_j^{(k+1)} - \hbar \frac{k+1}{2}}, & k = 0, \dots, \mathcal{M}-1, \\ A_k^*(u) &= \prod_{j=1}^{M^{(k)}} \frac{u - u_j^{*(k)} - \hbar \frac{k+2}{2}}{u - u_j^{*(k)} - \hbar \frac{k}{2}} \prod_{j=1}^{M^{(k+1)}} \frac{u - u_j^{*(k+1)} - \hbar \frac{k-1}{2}}{u - u_j^{*(k+1)} - \hbar \frac{k+1}{2}}, & k = \mathcal{M}, \dots, \mathcal{M} + \mathcal{N} - 1, \end{aligned}$$

and cancelations occur between dressing functions labeled by consecutive indices in expression (3.20). To fix the values of the  $\alpha_j^{(k)}$  and  $\beta_j^{(k)}$  for  $k \geq \mathcal{M}$  we start setting

$$\mathcal{T}'\mathcal{T} = \mathcal{T}'_{\mathcal{M}}(u + \hbar(\mathcal{M} - 1)) \cdots \mathcal{T}'_1(u) \mathcal{T}_{\mathcal{M}+\mathcal{N}}(u + \hbar(\mathcal{M} - 1)) \cdots \mathcal{T}_{\mathcal{M}+1}(u + \hbar(\mathcal{M} - \mathcal{N}))$$

and supertracing in all auxiliary spaces the identity

$$\mathcal{T}'\mathcal{T} = \text{Ber}^{-1}(u) A_{\mathcal{M}} A_{\mathcal{N}} + (1 - \Pi_{\mathcal{M}|\mathcal{N}}) \mathcal{T}'\mathcal{T} A_{\mathcal{M}} A_{\mathcal{N}} + \mathcal{T}'\mathcal{T} (1 - A_{\mathcal{M}} A_{\mathcal{N}}), \quad (3.21)$$

we get

$$st^*(u + \hbar(\mathcal{M} - 1)) \cdots st^*(u) st(u + \hbar(\mathcal{M} - 1)) \cdots st(u + \hbar(\mathcal{M} - \mathcal{N})) = (-1)^{\mathcal{N}} \text{Ber}^{-1}(u) + st_{\mathfrak{f}}^{(2)}(u),$$

where  $st_{\mathfrak{f}}^{(2)}(u) = str_{1 \dots \mathcal{M}+\mathcal{N}} [(1 - \Pi_{\mathcal{M}|\mathcal{N}}) \mathcal{T}'\mathcal{T} A_{\mathcal{M}} A_{\mathcal{N}} + \mathcal{T}'\mathcal{T} (1 - A_{\mathcal{M}} A_{\mathcal{N}})]$ . Acting again with the above equation on  $v$ , one obtains

$$\begin{aligned} &\Lambda^*(u + \hbar(\mathcal{M} - 1)) \cdots \Lambda^*(u) \Lambda(u + \hbar(\mathcal{M} - 1)) \cdots \Lambda(u + \hbar(\mathcal{M} - \mathcal{N})) = \\ &= (-1)^{\mathcal{N}} \prod_{l=1}^{\mathcal{M}} \lambda'_l(u + \hbar(\mathcal{M} - l)) \prod_{l=\mathcal{M}+1}^{\mathcal{M}+\mathcal{N}} \lambda_l(u + \hbar(2\mathcal{M} - l)) + \Lambda_{\mathfrak{f}}^{(2)}(u), \end{aligned} \quad (3.22)$$

where  $\Lambda_{\mathfrak{f}}^{(2)}(u) v = t_{\mathfrak{f}}^{(2)}(u) v$  and eq. (2.74) has been used. Picking up the term proportional to  $\lambda'_l(u + \hbar(\mathcal{M} - l)) \prod_{l=\mathcal{M}+1}^{\mathcal{M}+\mathcal{N}} \lambda_l(u + \hbar(2\mathcal{M} - l))$ , we get a second constraint on the dressing functions:

$$A_0^*(u + \hbar(\mathcal{M} - 1)) \cdots A_{\mathcal{M}-1}^*(u) A_{\mathcal{M}}(u + \hbar(\mathcal{M} - 1)) \cdots A_{\mathcal{M}+\mathcal{N}-1}(u + \hbar(\mathcal{M} - \mathcal{N})) = 1. \quad (3.23)$$

To satisfy this second constraint we set  $\alpha_j^{(k)} = u_j^{(k)} + \hbar(\mathcal{M} - \frac{k}{2} - 1)$ ,  $\beta_j^{(k+1)} = u_j^{(k+1)} + \hbar(\mathcal{M} - \frac{k-1}{2})$  for  $k = \mathcal{M}, \dots, \mathcal{M} + \mathcal{N} - 1$ , and  $\alpha_j^{*(k)} = u_j^{*(k)} + \hbar(\mathcal{M} - \frac{k}{2} - 1)$ ,  $\beta_j^{*(k+1)} = u_j^{*(k+1)} + \hbar(\mathcal{M} - \frac{k-1}{2})$  for  $k = 0, \dots, \mathcal{M} - 1$ , so that

$$A_k(u) = \prod_{j=1}^{M^{(k)}} \frac{u - u_j^{(k)} - \hbar(\mathcal{M} - \frac{k}{2} - 1)}{u - u_j^{(k)} - \hbar(\mathcal{M} - \frac{k}{2})} \prod_{j=1}^{M^{(k+1)}} \frac{u - u_j^{(k+1)} - \hbar(\mathcal{M} - \frac{k-1}{2})}{u - u_j^{(k+1)} - \hbar(\mathcal{M} - \frac{k+1}{2})}, \quad \mathcal{M} \leq k < \mathcal{M} + \mathcal{N},$$

$$A_k^*(u) = \prod_{j=1}^{M^{(k)}} \frac{u - u_j^{*(k)} - \hbar(\mathcal{M} - \frac{k}{2} - 1)}{u - u_j^{*(k)} - \hbar(\mathcal{M} - \frac{k}{2})} \prod_{j=1}^{M^{(k+1)}} \frac{u - u_j^{*(k+1)} - \hbar(\mathcal{M} - \frac{k-1}{2})}{u - u_j^{*(k+1)} - \hbar(\mathcal{M} - \frac{k+1}{2})}, \quad 0 \leq k < \mathcal{M}.$$

Again, it is seen that  $u_j^{*(\mathcal{M})} = u_j^{(\mathcal{M})} - \hbar\mathcal{M}$ .

**Remark 3.1** Relations (3.20) and (3.23) also hold when the  $A_l(u)$ ,  $A_l^*(u)$  functions are replaced with  $\widehat{A}_l(u)$ ,  $\widehat{A}_l^*(u)$ , thus leading to the same form for the dressing functions appearing in the eigenvalues (3.13) and (3.14).

**Remark 3.2** Using the  $c_k$  integers introduced in proposition 2.10, one can write a single expression for the dressing functions:

$$A_k(u) = \prod_{j=1}^{M^{(k)}} \frac{u - u_j^{(k)} - \frac{\hbar}{2}(c_{k+1} + (-1)^{[k+1]})}{u - u_j^{(k)} - \frac{\hbar}{2}c_k} \prod_{j=1}^{M^{(k+1)}} \frac{u - u_j^{(k+1)} - \frac{\hbar}{2}(c_k - (-1)^{[k+1]})}{u - u_j^{(k+1)} - \frac{\hbar}{2}c_{k+1}},$$

$$A_k^*(u) = \prod_{j=1}^{M^{(k)}} \frac{u - u_j^{*(k)} - \frac{\hbar}{2}(2\mathcal{M} - c_{k+1} - (-1)^{[k+1]})}{u - u_j^{*(k)} - \frac{\hbar}{2}(2\mathcal{M} - c_k)} \prod_{j=1}^{M^{(k)}} \frac{u - u_j^{*(k+1)} - \frac{\hbar}{2}(2\mathcal{M} - c_{k-1})}{u - u_j^{*(k+1)} - \frac{\hbar}{2}(2\mathcal{M} - c_{k+1})},$$

$$k = 1, \dots, \mathcal{M} + \mathcal{N} - 1, . \quad (3.24)$$

### 3.5 Bethe equations of closed spin chains

We have seen in the previous section that  $A_l(u) = \widehat{A}_l(u)$ , and that they have the form

$$A_l(u) = \prod_{k=1}^{M^{(l)}} \frac{u - u_k^{(l)} - \hbar \frac{l+2}{2}}{u - u_k^{(l)} - \hbar \frac{l}{2}} \prod_{k=1}^{M^{(l+1)}} \frac{u - u_k^{(l+1)} - \hbar \frac{l-1}{2}}{u - u_k^{(l+1)} - \hbar \frac{l+1}{2}}, \quad 0 \leq l < \mathcal{M},$$

$$A_l(u) = \prod_{k=1}^{M^{(l)}} \frac{u - u_k^{(l)} - \hbar(\mathcal{M} - \frac{l}{2} - 1)}{u - u_k^{(l)} - \hbar(\mathcal{M} - \frac{l}{2})} \prod_{k=1}^{M^{(l+1)}} \frac{u - u_k^{(l+1)} - \hbar(\mathcal{M} - \frac{l-1}{2})}{u - u_k^{(l+1)} - \hbar(\mathcal{M} - \frac{l+1}{2})}, \quad \mathcal{M} \leq l < \mathcal{M} + \mathcal{N},$$

with the convention  $M^{(0)} = M^{(\mathcal{M}+\mathcal{N})} = 0$ .

In order to establish analyticity of all eigenvalues of  $\Lambda(u)$  and of  $\widehat{\Lambda}(u)$ , one imposes that the residues of  $\Lambda(u)$  and  $\widehat{\Lambda}(u)$  at  $u = u_j^{(n)} + \hbar \frac{n}{2}$  for  $1 \leq j \leq M^{(n)}$ ,  $0 < n < \mathcal{M}$ , and at  $u = u_j^{(n)} + \hbar(\mathcal{M} - \frac{n}{2})$  for  $1 \leq j \leq M^{(n)}$ ,  $\mathcal{M} \leq n \leq \mathcal{M} + \mathcal{N} - 1$ , all vanish.

Introducing the function

$$\mathfrak{e}_n(u) \doteq \frac{u - \hbar \frac{n}{2}}{u + \hbar \frac{n}{2}}, \quad (3.25)$$



the vanishing of these residues leads to the following (Bethe ansatz) equations:

$$\prod_{k=1}^{M^{(n-1)}} \mathfrak{e}_{-1}(u_j^{(n)} - u_k^{(n-1)}) \prod_{k \neq j}^{M^{(n)}} \mathfrak{e}_2(u_j^{(n)} - u_k^{(n)}) \prod_{k=1}^{M^{(n+1)}} \mathfrak{e}_{-1}(u_j^{(n)} - u_k^{(n+1)}) = \frac{\lambda_n(u_j^{(n)} + \hbar \frac{n}{2})}{\lambda_{n+1}(u_j^{(n)} + \hbar \frac{n}{2})},$$

$$1 \leq j \leq M^{(n)}, \quad 0 < n < \mathcal{M} \quad (3.26)$$

$$\prod_{k=1}^{M^{(n-1)}} \mathfrak{e}_1(u_j^{(n)} - u_k^{(n-1)}) \prod_{k \neq j}^{M^{(n)}} \mathfrak{e}_{-2}(u_j^{(n)} - u_k^{(n)}) \prod_{k=1}^{M^{(n+1)}} \mathfrak{e}_1(u_j^{(n)} - u_k^{(n+1)}) = \frac{\lambda_n(u_j^{(n)} + \hbar(\mathcal{M} - \frac{n}{2}))}{\lambda_{n+1}(u_j^{(n)} + \hbar(\mathcal{M} - \frac{n}{2}))},$$

$$1 \leq j \leq M^{(n)}, \quad \mathcal{M} < n < \mathcal{M} + \mathcal{N} \quad (3.27)$$

$$\prod_{k=1}^{M^{(\mathcal{M}-1)}} \mathfrak{e}_{-1}(u_j^{(\mathcal{M})} - u_k^{(\mathcal{M}-1)}) \prod_{k=1}^{M^{(\mathcal{M}+1)}} \mathfrak{e}_1(u_j^{(\mathcal{M})} - u_k^{(\mathcal{M}+1)}) = \pm \frac{\lambda_{\mathcal{M}+1}(u_j^{(\mathcal{M})} + \hbar \frac{\mathcal{M}}{2})}{\lambda_{\mathcal{M}}(u_j^{(\mathcal{M})} + \hbar \frac{\mathcal{M}}{2})},$$

$$1 \leq j \leq M^{(\mathcal{M})} \quad (3.28)$$

where in the last equation the + sign (resp. - sign) corresponds to the  $\Lambda(u)$  BAE (resp.  $\hat{\Lambda}(u)$  BAE). One recognizes in the left-hand side of the BAEs the Cartan matrix of the  $gl(\mathcal{M}|\mathcal{N})$  superalgebra, while the right-hand side reflects the super-Yangian representation(s) spanned by the spin chain.

When the representations are finite dimensional, the right-hand side of these equations can be re-expressed in terms of Drinfeld polynomials. For instance, for the first set of BAEs, one gets

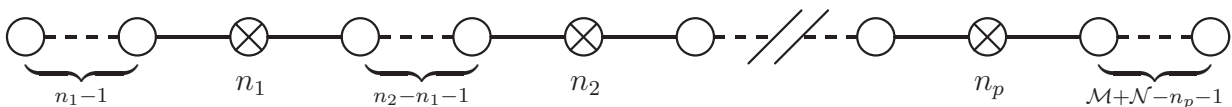
$$\frac{\lambda_n(u_j^{(n)} + \hbar \frac{n}{2})}{\lambda_{n+1}(u_j^{(n)} + \hbar \frac{n}{2})} = \frac{P_n(u_j^{(n)} + \hbar \frac{n}{2})}{P_{n+1}(u_j^{(n)} + \hbar \frac{n}{2})} \quad \text{where} \quad P_i(u) = \prod_{n=1}^L P_i^{[n]}(u), \quad (3.29)$$

$P_i^{[n]}(u)$  being the Drinfeld polynomials for each site.

## 4 Bethe equations for arbitrary Dynkin diagrams

As already mentioned, up-to-now we have worked with the distinguished Dynkin diagram and its associated gradation. However, several Dynkin diagrams can be used to describe the same superalgebra, leading to inequivalent Dynkin diagram, and thus to different presentations of the Bethe equations. For each of the grading (i.e. for each inequivalent Dynkin diagram), one can apply the above procedure to determine the form of the dressing functions. This has been noticed in [26] for open super-spin chains in the fundamental representation of  $sl(\mathcal{M}|\mathcal{N})$ . We generalize it for arbitrary super-spin chains. The dressing functions keep essentially the same structure, with the following rules.

The inequivalent Dynkin diagrams of the  $sl(\mathcal{M}|\mathcal{N})$  superalgebras contain only bosonic roots of same square length ("white dots"), normalized to 2, and isotropic fermionic roots ("grey dots"), which square to zero. A given diagram is completely characterized by the  $p$ -uple of integers  $0 < n_1 < \dots < n_p < \mathcal{M} + \mathcal{N}$  labelling the positions of the grey dots of the diagram:



where the total number of (grey and white) dots is  $\mathcal{M} + \mathcal{N} - 1$ . Formally, we define  $n_0 = 0$  and  $n_{p+1} = \mathcal{M} + \mathcal{N}$  although there is actually no root at these positions. Such a diagram defined by the  $p$ -uple  $(n_i)_{i=1\dots p}$  corresponds to the superalgebra  $sl(\mathcal{M}|\mathcal{N})$  with

$$\mathcal{M} = \sum_{\substack{i \text{ odd} \\ i \leq p+1}} n_i - \sum_{\substack{i \text{ even} \\ i < p+1}} n_i \quad \text{and} \quad \mathcal{N} = \sum_{\substack{i \text{ even} \\ i \leq p+1}} n_i - \sum_{\substack{i \text{ odd} \\ i < p+1}} n_i. \quad (4.1)$$

Accordingly, the  $\mathbb{Z}_2$ -grading is defined by

$$[j] = \frac{1 - (-1)^k}{2}, \quad \text{i.e.} \quad (-1)^{[j]} = (-1)^k, \quad \text{for} \quad n_k + 1 \leq j \leq n_{k+1}, \quad 0 \leq k \leq \mathcal{M} + \mathcal{N}. \quad (4.2)$$

For each of these gradings, one can compute a new value for the parameters

$$c_k = \sum_{j=1}^k (-1)^{[j]}, \quad k = 1, \dots, \mathcal{M} + \mathcal{N}. \quad (4.3)$$

Then, the dressing functions will keep the same form (3.24), but with now the above value for the parameters  $c_k$ . Computing the residues for  $\Lambda(u)$  with these new dressing functions, leads to the Bethe equations

$$(1 - (-1)^{[l]} \langle \alpha_\ell, \alpha_\ell \rangle) \prod_{k=1}^{\mathcal{M} + \mathcal{N} - 1} \prod_{j=1}^{M^{(k)}} \mathbf{e}_{\langle \alpha_\ell, \alpha_k \rangle} (u_i^{(\ell)} - u_j^{(k)}) = \frac{\lambda_\ell (u_i^{(\ell)} + \frac{\hbar}{2} c_\ell)}{\lambda_{\ell+1} (u_i^{(\ell)} + \frac{\hbar}{2} c_\ell)},$$

$$i = 1, \dots, M^{(\ell)}, \quad 1 \leq \ell < \mathcal{M} + \mathcal{N} - 1. \quad (4.4)$$

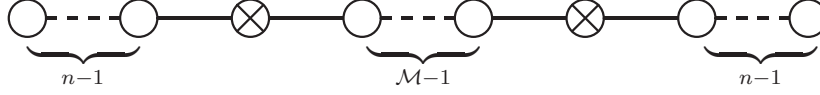
where  $\langle \alpha_\ell, \alpha_k \rangle$  is the scalar product of the simple roots, numbered *as they are ordered by the chosen Dynkin diagram*. This single set of equations describe *all* the Bethe equations, whatever the gradation (the Dynkin diagram) is, and whatever the representations on each site of the spin chain are. In the particular case of only (mixture of) fundamental representation and/or its contragredient on all sites, we recover the isotropic limit ( $q \rightarrow 1$ ) of the spectrum and BAE computed in [27]. These equations are also equivalent to the ones presented in [16], the different gradations here being related to the different possible paths (forms of the ‘hook’) in [16].

Explicitely, in  $sl(\mathcal{M}|\mathcal{N})$ , denoting  $\alpha_j$  the simple roots, that we label according to their position  $j = 1, \dots, \mathcal{M} + \mathcal{N}$  in the Dynkin diagram, their norm is given by  $\langle \alpha_j, \alpha_j \rangle = (-1)^{[j]} 2$  for the bosonic ‘white’ roots and by  $\langle \alpha_j, \alpha_j \rangle = 0$  for the fermionic ‘grey’ roots. Moreover, the scalar products between different simple roots are all zero *but* for the simple roots which are linked in the Dynkin diagram. Linked roots have scalar product  $\langle \alpha_j, \alpha_{j+1} \rangle = -(-1)^{[j+1]}$ . For more informations on the construction of simple roots and Dynkin diagrams for superalgebras, see e.g. [20].

It should be clear that, since the different presentations (i.e. Dynkin diagrams) describe the same superalgebra and the same representations on the chain, the spectrum will be identical, although the dressing functions and the BAE look different.

## 4.1 Bethe equations for the symmetric grading

In the case of  $sl(\mathcal{M}|2n)$ , there exists a symmetric Dynkin diagram with two isotropic fermionic simple roots in positions  $n$  and  $\mathcal{M} + n$ :



We give here the explicit expression for the dressing functions and Bethe Ansatz equations for this diagram, thus taking  $\mathcal{N} = 2n$ , and ordering the indices as in eq.(2.8):

$$[i] = \begin{cases} 0, & 1 \leq i \leq n \quad \text{and} \quad \mathcal{M} + n + 1 \leq i \leq \mathcal{M} + \mathcal{N}, \\ 1, & n + 1 \leq i \leq \mathcal{M} + n. \end{cases},$$

i.e.

$$c_k = \begin{cases} k, & k \leq n, \\ \mathcal{N} - k, & n < k \leq \mathcal{M} + n, \\ k - 2\mathcal{M}, & \mathcal{M} + n < k \leq \mathcal{M} + \mathcal{N}. \end{cases} \quad (4.5)$$

This choice of the grading implies that the elements of  $T^{(\mathcal{M})}(u)$  (resp.  $T^{(\mathcal{N})}(u)$ ) generate now a  $\mathcal{Y}_{-\hbar}(\mathcal{M})$  (resp.  $\mathcal{Y}_{\hbar}(\mathcal{N})$ ) bosonic subalgebra. As a consequence, the expressions for the quantum Berezinian and its inverse are modified as follows:

$$Ber(u) = \text{qdet } T^{(\mathcal{N})}(u - \hbar(\mathcal{M} - \mathcal{N} + 1)) \text{qdet } T^{*(\mathcal{M})}(u - \hbar\mathcal{M}),$$

$$Ber^{-1}(u) = \text{qdet } T^{*(\mathcal{N})}(u + \hbar(\mathcal{N} - 1)) \text{qdet } T^{(\mathcal{M})}(u - \hbar(\mathcal{M} - \mathcal{N})).$$

To determine its value on  $v^+$  we rewrite the quantum Berezinian for the symmetric Dynkin diagram case as

$$\begin{aligned} Ber(u) &= \sum_{\sigma \in S_{\mathcal{N}}} \text{sgn}(\sigma) T_{\sigma(1),1}(u - \hbar(\mathcal{M} - \mathcal{N} + 1)) \cdots T_{\sigma(n),n}(u - \hbar(\mathcal{M} - n)) \times \\ &\times T_{\mathcal{M}+\sigma(n+1),\mathcal{M}+n+1}(u - \hbar(\mathcal{M} - n + 1)) \cdots T_{\mathcal{M}+\sigma(\mathcal{N}),\mathcal{M}+\mathcal{N}}(u - \hbar\mathcal{M}) \times \\ &\times \sum_{\tau \in S_{\mathcal{M}}} \text{sgn}(\tau) T_{n+\tau(1),n+1}^*(u - \hbar\mathcal{M}) \cdots T_{n+\tau(\mathcal{M}),n+\mathcal{M}}^*(u - \hbar), \end{aligned}$$

obtaining:

$$Ber(u) v^+ = \prod_{l=1}^n \lambda_l(u - \hbar(\mathcal{M} - l + 1)) \prod_{l=n+1}^{\mathcal{M}+n} \lambda_l^*(u - \hbar(\mathcal{M} - l + n + 1)) \prod_{l=\mathcal{M}+n+1}^{\mathcal{M}+\mathcal{N}} \lambda_l(u - \hbar(2\mathcal{M} - l + 1)) v^+$$

In the same way we can compute the constant value of  $Ber^{-1}(u)$  on the  $v^+$  module. Since

$$\begin{aligned} Ber^{-1}(u) &= \sum_{\sigma \in S_{\mathcal{N}}} \text{sgn}(\sigma) T_{\sigma(1),1}^*(u + \hbar(\mathcal{N} - 1)) \cdots T_{\sigma(n),n}^*(u + \hbar n) \times \\ &\times T_{\mathcal{M}+\sigma(n+1),\mathcal{M}+n+1}^*(u + \hbar(n - 1)) \cdots T_{\mathcal{M}+\sigma(\mathcal{N}),\mathcal{M}+\mathcal{N}}^*(u) \times \\ &\times \sum_{\tau \in S_{\mathcal{M}}} \text{sgn}(\tau) T_{n+\tau(1),n+1}(u - \hbar(\mathcal{M} - \mathcal{N})) \cdots T_{n+\tau(\mathcal{M}),n+\mathcal{M}}(u + \hbar(\mathcal{N} - 1)), \end{aligned}$$

we get

$$Ber^{-1}(u) v^+ = \prod_{l=1}^n \lambda_l^*(u + \hbar(\mathcal{N} - l)) \prod_{l=n+1}^{\mathcal{M}+n} \lambda_l(u - \hbar(\mathcal{N} - l + n)) \prod_{l=\mathcal{M}+n+1}^{\mathcal{M}+\mathcal{N}} \lambda_l^*(u + \hbar(\mathcal{M} + \mathcal{N} - l)) v^+.$$

The steps leading to the dressing functions (3.24) can now be repeated taking into account the different form of the value of the quantum Berezinian: in particular, one can show that the constraints (3.20) and (3.23) are to be replaced with:

$$A_0(u) \cdots A_{n-1}(u + \hbar(n-1)) A_n^*(u) \cdots A_{\mathcal{M}+n-1}^*(u + \hbar(\mathcal{M}-1)) \times \\ \times A_{\mathcal{M}+n}(u + \hbar n) \cdots A_{\mathcal{M}+\mathcal{N}-1}(u + \hbar(\mathcal{N}-1)) = 1, \quad (4.6)$$

and

$$A_0^*(u + \hbar(\mathcal{N}-1)) \cdots A_{n-1}^*(u + \hbar n) A_n(u + \hbar(\mathcal{N}-1)) \cdots A_{\mathcal{M}+n-1}(u + \hbar(\mathcal{N}-\mathcal{M})) \times \\ \times A_{\mathcal{M}+n}^*(u + \hbar(n-1)) \cdots A_{\mathcal{M}+\mathcal{N}-1}^*(u) = 1. \quad (4.7)$$

Both these constraints are satisfied by the dressing functions (3.24). As a general rule, at each fermionic root two dressing functions  $A$  and  $A^*$  meet, and the  $u_j^{(k)}$  parameters must satisfy an additional relation<sup>3</sup> of the form  $u_j^{*(k)} = u_j^{(k)} - \hbar\mathcal{M}$ . We are now in position to write the Bethe Ansatz equations for the symmetric Dynkin diagram, requiring the transfer matrix eigenvalue

$$\Lambda(u) = \sum_{k=1}^{\mathcal{M}+\mathcal{N}} (-1)^{[k]} A_{k-1}(u) \lambda_k(u)$$

to have vanishing residues at  $u = u_j^{(l)} + \frac{\hbar}{2}c_l$  for  $l = 1, \dots, \mathcal{M} + \mathcal{N} - 1$  and  $j = 1, \dots, M^{(l)}$ . The BAEs take the form:

$$\prod_{k=1}^{M^{(l-1)}} \mathbf{e}_{-1}(u_j^{(l)} - u_k^{(l-1)}) \prod_{k \neq j}^{M^{(l)}} \mathbf{e}_2(u_j^{(l)} - u_k^{(l)}) \prod_{k=1}^{M^{(l+1)}} \mathbf{e}_{-1}(u_j^{(l)} - u_k^{(l+1)}) = \frac{\lambda_{l+1}(u_j^{(l)} + \frac{\hbar}{2}c_l)}{\lambda_l(u_j^{(l)} + \frac{\hbar}{2}c_l)},$$

$$1 \leq j \leq M^{(l)}, \quad 1 \leq l < n \quad \text{and} \quad \mathcal{M} + n + 1 < l < \mathcal{M} + \mathcal{N}$$

$$\prod_{k=1}^{M^{(n-1)}} \mathbf{e}_{-1}(u_j^{(n)} - u_k^{(n-1)}) \prod_{k=1}^{M^{(n+1)}} \mathbf{e}_1(u_j^{(n)} - u_k^{(n+1)}) = \frac{\lambda_{n+1}(u_j^{(n)} + \frac{\hbar}{2}n)}{\lambda_n(u_j^{(n)} + \frac{\hbar}{2}n)},$$

$$\prod_{k=1}^{M^{(l-1)}} \mathbf{e}_1(u_j^{(l)} - u_k^{(l-1)}) \prod_{k \neq j}^{M^{(l)}} \mathbf{e}_{-2}(u_j^{(l)} - u_k^{(l)}) \prod_{k=1}^{M^{(l+1)}} \mathbf{e}_1(u_j^{(l)} - u_k^{(l+1)}) = \frac{\lambda_{l+1}(u_j^{(l)} + \hbar(\mathcal{M} - \frac{l}{2}))}{\lambda_l(u_j^{(l)} + \hbar(\mathcal{M} - \frac{l}{2}))},$$

$$1 \leq j \leq M^{(l)}, \quad n < l < \mathcal{M} + n$$

$$\prod_{k=1}^{M^{(\mathcal{M}+n-1)}} \mathbf{e}_1(u_j^{(\mathcal{M}+n)} - u_k^{(\mathcal{M}+n-1)}) \prod_{k=1}^{M^{(\mathcal{M}+n+1)}} \mathbf{e}_{-1}(u_j^{(\mathcal{M}+n)} - u_k^{(\mathcal{M}+n+1)}) = \frac{\lambda_{\mathcal{M}+n+1}(u_j^{(\mathcal{M}+n)} + \frac{\hbar}{2}(n - \mathcal{M}))}{\lambda_{\mathcal{M}+n}(u_j^{(\mathcal{M}+n)} + \frac{\hbar}{2}(n - \mathcal{M}))},$$

in agreement with eq.(4.4).

---

<sup>3</sup>In the distinguished Dynkin diagram case there is only one fermionic root, corresponding to the  $u_j^{*(\mathcal{M})} = u_j^{(\mathcal{M})} - \hbar\mathcal{M}$  relation obtained in the previous section.

## 5 Open super-spin chains

### 5.1 Open chains monodromy and transfer matrices

As in the closed chain case the supercommutation relations defining the reflection algebra allow us to show that the transfer matrix

$$b(u) = \text{str} \left( K^+(u) B(u) \right) = \sum_{k,l=1}^{\mathcal{M}+\mathcal{N}} (-1)^{[k]} K_{kl}^+(u) B_{lk}(u). \quad (5.1)$$

generates a commutative family

$$[b(u), b(v)] = 0,$$

provided the matrix  $K^+(u)$  obeys the ‘dual’ reflection equation:

$$\begin{aligned} R_{12}(-u+v) K_1^+(u)^t R_{21}(-u-v-\hbar(\mathcal{M}-\mathcal{N})) K_2^+(v)^t = \\ K_2^+(v)^t R_{12}(-u-v-\hbar(\mathcal{M}-\mathcal{N})) K_1^+(u)^t R_{21}(-u+v). \end{aligned} \quad (5.2)$$

The classification of such matrices is deduced from the proposition 2.9. Indeed, if  $K^+(u)$  obeys the dual reflection equation (5.2), then  $K^+(-u-\hbar\rho)^t$ , with  $\rho = \mathcal{M}-\mathcal{N}$ , obeys reflection equation (2.78), so that  $K^+(u) = U' \left( \mathbb{E}' + \frac{\xi'}{u} \mathbb{I} \right) U'^{-1}$  for some new parameters  $U'$ ,  $\mathbb{E}'$  and  $\xi'$ .

We further assume that the matrix  $K^+(u)$  commute with the matrix  $K^-(v)$ . Then, all the  $K^pm(u)$  matrices are diagonalizable by the same matrix  $U$ , independent of the spectral parameter. Thus, one can assume that  $K^+(u)$  is also diagonal and analytic:

$$K^+(u) = \text{diag} \left( \underbrace{\xi' - u, \dots, \xi' - u}_{L'_1}, \underbrace{u + \xi', \dots, u + \xi'}_{L'_2 - L'_1}, \xi' - u, \dots, \xi' - u \right). \quad (5.3)$$

Again, upon representation, one constructs a monodromy matrix  $\mathcal{B}(u)$  for the  $L$  sites open chain. In order to get analytical entries for the transfer matrix, we adopt the normalization (3.11) for  $T(u)$  and  $\mathcal{T}(u)$ , and define:

$$\mathcal{B}(u) = (\mathcal{T}(u) \otimes \dots \otimes \mathcal{T}(u)) K(u) (\mathcal{T}^{-1}(-u) \otimes \dots \otimes \mathcal{T}^{-1}(-u)). \quad (5.4)$$

$$\widehat{b}(u) = \text{str} \left( K^+(u) \mathcal{B}(u) \right) = \sum_{k=1}^{\mathcal{M}+\mathcal{N}} (-1)^{[k]} K_{kk}^+(u) \mathcal{B}_{kk}(u). \quad (5.5)$$

### 5.2 Symmetry of transfer matrices

As we did in section 3.2 for the closed chain case, we now turn to determine the symmetry of the model whose transfer matrix is given by (5.1). Without any loss of generality we assume in what follows that  $L_1 < \mathcal{M} < L_2$ .

**Proposition 5.1** *We consider the transfer matrix  $b(u)$  describing open spin chain models with boundary conditions given by  $K(u)$  and  $K^+(u)$ , see eq. (2.83) and (5.3), with  $L_1, L'_1 < \mathcal{M}$  and  $L_2, L'_2 > \mathcal{M}$ . Let*

$$\mathfrak{m}_j = \min(L_j, L'_j) \quad \text{and} \quad \mathfrak{M}_j = \max(L_j, L'_j), \quad j = 1, 2.$$

Then,  $b(u)$  admits a  $gl(\mathfrak{m}_1|\mathcal{M} + \mathcal{N} - \mathfrak{m}_2) \oplus \mathcal{G} \oplus gl(\mathcal{M} - \mathfrak{M}_1|\mathfrak{M}_2 - \mathcal{M})$  symmetry, where

$$\mathcal{G} = \begin{cases} gl(\mathfrak{M}_1 - \mathfrak{m}_1) \oplus gl(\mathfrak{M}_2 - \mathfrak{m}_2), & \text{if } (\mathfrak{m}_1, \mathfrak{m}_2) = (L_1, L_2) \text{ or } (\mathfrak{m}_1, \mathfrak{m}_2) = (L'_1, L'_2), \\ gl(\mathfrak{M}_1 - \mathfrak{m}_1|\mathfrak{M}_2 - \mathfrak{m}_2) & \text{otherwise.} \end{cases}$$

Proof: Supertracing in the first auxiliary space the supercommutation relations (2.80), and expanding them in  $u$  and  $v$ , one reads, from the  $v^1$  order term

$$\left[ b(u), B_{ij}^{(1)} \right] = -B_{ij}(u) (K_{ii}^+(u) - K_{jj}^+(u)) (\theta_i + \theta_j). \quad (5.6)$$

Since  $B_{ij}^{(1)} = 0$  when  $\theta_i + \theta_j = 0$  (see eq. 2.82), one deduces that a non-zero generator  $B_{ij}^{(1)}$  commutes with  $b(u)$  if and only if  $K_{ii}^+(u) = K_{jj}^+(u)$ , that is to say  $\theta'_i = \theta'_j$ . The symmetry (super)algebra is thus generated by the elements of  $gl(L_1|\mathcal{M} + \mathcal{N} - L_2) \oplus gl(\mathcal{M} - L_1|L_2 - \mathcal{M})$  obeying this relation: an enumeration of them ends the proof. ■

### 5.3 Pseudovacuum for open chain transfer matrices

A direct computation, using the result of propositions 2.10 and 2.5, shows that the highest weight vector  $v^+$  is an eigenvector of  $\widehat{b}(u)$ :

$$\begin{aligned} \widehat{b}(u) v^+ &= \sum_{j=1}^{\mathcal{M}+\mathcal{N}} (-1)^{[j]} K_{jj}^+(u) \mathcal{B}_{jj}(u) v^+ = \Lambda_0(u) v^+, \\ \Lambda_0(u) &= \sum_{j=1}^{\mathcal{M}+\mathcal{N}} (-1)^{[j]} g_j(u) \beta_j(u). \end{aligned}$$

Here the functions  $g_j(u)$ ,  $j = 1, \dots, \mathcal{M} + \mathcal{N}$ , depend only on the boundary matrix, while the functions  $\beta_j(u)$  are determined by the representations on the chain:

$$g_j(u) = \frac{2u(2u - \hbar(\mathcal{M} - \mathcal{N}))}{(2u - \hbar c_{j-1})(2u - \hbar c_j)} K_{jj}(u) K_{jj}^+(u) \quad (5.7)$$

$$\beta_j(u) = \left( \prod_{m=1}^{j-1} \lambda_m(-u + \hbar c_m) \right) \lambda_j(u) \left( \prod_{m=j+1}^{\mathcal{M}+\mathcal{N}} \lambda_m(-u + \hbar c_{m-1}) \right). \quad (5.8)$$

In the above expressions the  $\lambda_k(u)$ 's are again the products of the eigenvalues for each site of the chain, as in (3.12).

### 5.4 Dressing functions for open chains

We assume that all the eigenvalues of  $b(u)$  can be written as

$$\Lambda(u) = \sum_{k=1}^{\mathcal{M}+\mathcal{N}} (-1)^{[k]} g_k(u) \beta_k(u) A_{k-1}(u), \quad (5.9)$$

with  $g_k(u)$  and  $\beta_k(u)$  given by (5.7) and (5.8) respectively, and dressing functions  $A_k(u)$  to be determined. The vanishing of the residues of  $\Lambda(u)$  at  $u = \frac{\hbar}{2}c_k$  implies that

$$A_{k-1}(\frac{\hbar}{2}c_k) = A_k(\frac{\hbar}{2}c_k), \quad \text{for } 1 \leq k \leq \mathcal{M} - \mathcal{N} - 1.$$

Using expression (3.15) for the dressing functions one can show that the  $M^{(k)}$ 's are even and that the simplest non-trivial way to satisfy the above constraint is to set

$$\begin{aligned} A_k(u) &= \prod_{j=1}^{M^{(k)}} \frac{u - u_j^{(k)} - \frac{\hbar}{2}(c_{k+1} + (-1)^{[k+1]})}{u - u_j^{(k)} - \frac{\hbar}{2}c_k} \frac{u + u_j^{(k)} - \frac{\hbar}{2}(c_{k+1} + (-1)^{[k+1]})}{u + u_j^{(k)} - \frac{\hbar}{2}c_k} \\ &\times \prod_{j=1}^{M^{(k+1)}} \frac{u - u_j^{(k+1)} - \frac{\hbar}{2}(c_k - (-1)^{[k+1]})}{u - u_j^{(k+1)} - \frac{\hbar}{2}c_{k+1}} \frac{u + u_j^{(k+1)} - \frac{\hbar}{2}(c_k - (-1)^{[k+1]})}{u + u_j^{(k+1)} - \frac{\hbar}{2}c_{k+1}}, \end{aligned}$$

for  $k = 0, \dots, \mathcal{M} + \mathcal{N} - 1$ , with the usual convention  $M^{(0)} = M^{(\mathcal{M}+\mathcal{N})} = 0$ .

## 5.5 Bethe equations for the open chain

In order to establish analyticity of all eigenvalues, one imposes that the residues of  $\Lambda(u)$  at  $u = u_n^{(l)} + \frac{\hbar}{2}c_l$ , for  $1 \leq n \leq M^{(l)}$ ,  $0 < l \leq \mathcal{M} + \mathcal{N} - 1$ , all vanish. Using the definition (3.25) for the  $\mathfrak{e}_n(u)$  function one has the following set of Bethe Ansatz equation:

$$\begin{aligned} &\prod_{j \neq n}^{M^{(l)}} \mathfrak{e}_2(u_n^{(l)} - u_j^{(l)}) \prod_{j=1}^{M^{(l)}} \mathfrak{e}_2(u_n^{(l)} + u_j^{(l)}) \prod_{\tau=\pm 1} \prod_{j=1}^{M^{(l+\tau)}} \mathfrak{e}_{-1}(u_n^{(l)} - u_j^{(l+\tau)}) \mathfrak{e}_{-1}(u_n^{(l)} + u_j^{(l+\tau)}) = \\ &= \frac{\beta_l(u_n^{(l)} + \frac{\hbar}{2}c_l)}{\beta_{l+1}(u_n^{(l)} + \frac{\hbar}{2}c_l)} \frac{g_l(u_n^{(l)} + \frac{\hbar}{2}c_l)}{g_{l+1}(u_n^{(l)} + \frac{\hbar}{2}c_l)}, \quad l = 1 \leq l < \mathcal{M}, \\ &\prod_{j=1}^{M^{(\mathcal{M}+1)}} \mathfrak{e}_1(u_n^{(\mathcal{M})} - u_j^{(\mathcal{M}+1)}) \mathfrak{e}_1(u_n^{(\mathcal{M})} + u_j^{(\mathcal{M}+1)}) \prod_{j=1}^{M^{(\mathcal{M}-1)}} \mathfrak{e}_{-1}(u_n^{(\mathcal{M})} - u_j^{(\mathcal{M}-1)}) \mathfrak{e}_{-1}(u_n^{(\mathcal{M})} + u_j^{(\mathcal{M}-1)}) = \\ &= \frac{\beta_{\mathcal{M}}(u_n^{(\mathcal{M})} + \frac{\hbar}{2}\mathcal{M})}{\beta_{\mathcal{M}+1}(u_n^{(\mathcal{M})} + \frac{\hbar}{2}\mathcal{M})} \frac{g_{\mathcal{M}}(u_n^{(\mathcal{M})} + \frac{\hbar}{2}c_{\mathcal{M}})}{g_{\mathcal{M}+1}(u_n^{(\mathcal{M})} + \frac{\hbar}{2}c_{\mathcal{M}})}, \quad l = \mathcal{M}, \\ &\prod_{j \neq n}^{M^{(l)}} \mathfrak{e}_{-2}(u_n^{(l)} - u_j^{(l)}) \prod_{j=1}^{M^{(l)}} \mathfrak{e}_{-2}(u_n^{(l)} + u_j^{(l)}) \prod_{\tau=\pm 1} \prod_{j=1}^{M^{(l+\tau)}} \mathfrak{e}_1(u_n^{(l)} - u_j^{(l+\tau)}) \mathfrak{e}_1(u_n^{(l)} + u_j^{(l+\tau)}) = \\ &= \frac{\beta_l(u_n^{(l)} + \frac{\hbar}{2}c_l)}{\beta_{l+1}(u_n^{(l)} + \frac{\hbar}{2}c_l)} \frac{g_l(u_n^{(l)} + \frac{\hbar}{2}c_l)}{g_{l+1}(u_n^{(l)} + \frac{\hbar}{2}c_l)}, \quad l = \mathcal{M} < l < \mathcal{M} + \mathcal{N}. \end{aligned} \tag{5.10}$$

As in the closed case, the left hand side of the Bethe equations only depends on the choice of the algebra, while the right hand side explicitly depends on the choice of the representation (through the  $\beta_l(u)$ 's functions, eq. (5.8)) and on the reflection matrix (through the  $g_l(u)$ 's functions, eq. (5.7)).

## 5.6 Bethe equations for other Dynkin diagrams

We turn now to the calculation of the spectrum and Bethe equations of open super-spin chains for other Dynkin diagrams. The rules will be the same as the ones given for the closed case (see section 4). The functions  $g_k(u)$  have a form similar to (5.7), with a change of increasing or decreasing behaviour of the poles each time a grey (fermionic) root is met, due to the change in the definition of the  $\mathbb{Z}_2$ -grading, and thus in the parameters  $c_k$ , as given in (4.3).

The Bethe Ansatz equations read, for  $\ell = 1, \dots, \mathcal{M} + \mathcal{N} - 1$  and  $i = 1, \dots, M^{(\ell)}$

$$\epsilon_\ell \prod_{k=1}^{\mathcal{M}+\mathcal{N}-1} \prod_{j=1}^{M^{(k)}} \mathfrak{e}_{\langle \alpha_\ell, \alpha_k \rangle}(u_i^{(\ell)} - u_j^{(k)}) \mathfrak{e}_{\langle \alpha_\ell, \alpha_k \rangle}(u_i^{(\ell)} + u_j^{(k)}) = \frac{\beta_\ell(u_i^{(\ell)} + \frac{\hbar}{2} c_\ell)}{\beta_{\ell+1}(u_i^{(\ell)} + \frac{\hbar}{2} c_\ell)} \frac{g_\ell(u_i^{(\ell)} + \frac{\hbar}{2} c_\ell)}{g_{\ell+1}(u_i^{(\ell)} + \frac{\hbar}{2} c_\ell)},$$

where  $\epsilon_\ell = (1 - (-1)^{|\ell|} \langle \alpha_\ell, \alpha_\ell \rangle)$ , as in the closed spin chain case. As an example, we specialize the above formulas to the symmetric Dynkin diagram case. The  $g$  functions are in this case:

$$g_l(u) = \frac{u(u + \frac{\hbar(\mathcal{M}-\mathcal{N})}{2})}{(u + \frac{\hbar(l-1)}{2})(u + \frac{\hbar l}{2})}, \quad l = 1, \dots, \mathcal{N}/2, \quad (5.11)$$

$$g_l(u) = \frac{u(u + \frac{\hbar(\mathcal{M}-\mathcal{N})}{2})}{(u + \frac{\hbar(\mathcal{N}-l+1)}{2})(u + \frac{\hbar(\mathcal{N}-l)}{2})}, \quad l = \mathcal{N}/2 + 1, \dots, \mathcal{M} + \mathcal{N}/2, \quad (5.12)$$

$$g_l(u) = \frac{u(u + \frac{\hbar(\mathcal{M}-\mathcal{N})}{2})}{(u + \frac{\hbar(l-2\mathcal{M}-1)}{2})(u + \frac{\hbar(l-2\mathcal{M})}{2})}, \quad l = \mathcal{M} + \mathcal{N}/2 + 1, \dots, \mathcal{M} + \mathcal{N}. \quad (5.13)$$

The Bethe equations, obtained by imposing analyticity of  $\Lambda(u)$  at points  $u = u_k^{(l)} + \hbar c_l/2$ , for  $1 \leq k \leq M^{(l)}$  and  $l = 1, \dots, \mathcal{M} + \mathcal{N} - 1$ , are:

$$\begin{aligned} & \prod_{j \neq k}^{M^{(l)}} \mathfrak{e}_2(u_k^{(l)} - u_j^{(l)}) \prod_{j=1}^{M^{(l)}} \mathfrak{e}_2(u_k^{(l)} + u_j^{(l)}) \prod_{\tau=\pm 1} \prod_{j=1}^{M^{(l+\tau)}} \mathfrak{e}_{-1}(u_k^{(l)} - u_j^{(l+\tau)}) \mathfrak{e}_{-1}(u_k^{(l)} + u_j^{(l+\tau)}) = \\ & = \frac{\beta_l(u_k^{(l)} + \frac{\hbar}{2} c_l)}{\beta_{l+1}(u_k^{(l)} + \frac{\hbar}{2} c_l)} \frac{g_l(u_k^{(l)} + \frac{\hbar}{2} c_l)}{g_{l+1}(u_k^{(l)} + \frac{\hbar}{2} c_l)}, \quad 1 \leq l < n \text{ and } \mathcal{M} + n < l < \mathcal{M} + \mathcal{N}, \end{aligned} \quad (5.14)$$

$$\begin{aligned} & \prod_{j=1}^{M^{(n-1)}} \mathfrak{e}_{-1}(u_k^{(n)} - u_j^{(n-1)}) \mathfrak{e}_{-1}(u_k^{(n)} + u_j^{(n-1)}) \prod_{j=1}^{M^{(n+1)}} \mathfrak{e}_1(u_k^{(n)} - u_j^{(n+1)}) \mathfrak{e}_1(u_k^{(n)} + u_j^{(n+1)}) = \\ & = \frac{\beta_n(u_k^{(n)} + \frac{\hbar}{2} c_n)}{\beta_{n+1}(u_k^{(n)} + \frac{\hbar}{2} c_n)} \frac{g_n(u_k^{(n)} + \frac{\hbar}{2} c_n)}{g_{n+1}(u_k^{(n)} + \frac{\hbar}{2} c_n)}, \quad l = n, \end{aligned} \quad (5.15)$$

$$\begin{aligned} & \prod_{j \neq k}^{M^{(l)}} \mathfrak{e}_{-2}(u_k^{(l)} - u_j^{(l)}) \prod_{j=1}^{M^{(l)}} \mathfrak{e}_{-2}(u_k^{(l)} + u_j^{(l)}) \prod_{\tau=\pm 1} \prod_{j=1}^{M^{(l+\tau)}} \mathfrak{e}_1(u_k^{(l)} - u_j^{(l+\tau)}) \mathfrak{e}_1(u_k^{(l)} + u_j^{(l+\tau)}) = \\ & = \frac{\beta_l(u_k^{(l)} + \frac{\hbar}{2} c_l)}{\beta_{l+1}(u_k^{(l)} + \frac{\hbar}{2} c_l)} \frac{g_l(u_k^{(l)} + \frac{\hbar}{2} c_l)}{g_{l+1}(u_k^{(l)} + \frac{\hbar}{2} c_l)}, \quad n < l < \mathcal{M} + n, \end{aligned} \quad (5.16)$$



$$\begin{aligned}
& \prod_{j=1}^{M^{(l-1)}} \mathbf{e}_1(u_k^{(l)} - u_j^{(l-1)}) \mathbf{e}_1(u_k^{(l)} + u_j^{(l-1)}) \prod_{j=1}^{M^{(l+1)}} \mathbf{e}_{-1}(u_k^{(l)} - u_j^{(l+1)}) \mathbf{e}_{-1}(u_k^{(l)} + u_j^{(l+1)}) = \\
& = \frac{\beta_l(u_k^{(l)} + \frac{\hbar}{2} c_l)}{\beta_{l+1}(u_k^{(l)} + \frac{\hbar}{2} c_l)} \frac{g_l(u_k^{(l)} + \frac{\hbar}{2} c_l)}{g_{l+1}(u_k^{(l)} + \frac{\hbar}{2} c_l)}, \quad l = \mathcal{M} + n.
\end{aligned} \tag{5.17}$$

## 6 Examples

In this section we discuss the application of our approach to few examples. We will replace the  $\hbar$  parameter with the imaginary unit  $-i$ , as it is customary in dealing with spin chains.

Let us stress that, although in all examples, the energies will look identical (up to an irrelevant additive constant), the spectrum and Hamiltonians are indeed different. In fact, the energies are functions of the Bethe roots, which obey different Bethe equations, specified by the representations entering the spin chain.

### 6.1 Closed super-spin chain in the fundamental representation

Choosing for each site of the closed chain the fundamental representation, we get the usual supersymmetric spin chains. In the fundamental representation, one has  $\mu_i^{[n]} = \delta_{i,1}$  for all sites  $n = 1, \dots, L$ , so that the eigenvalues (3.12) become:

$$\lambda_k(u) = \begin{cases} \prod_{n=1}^L (u - a_n - i) & k = 1, \\ \prod_{n=1}^L (u - a_n) & k \neq 1. \end{cases} \tag{6.1}$$

Plugging these expressions in the Bethe equations of section 3.5, we get

$$\begin{aligned}
& \prod_{k=1}^{M^{(n-1)}} \mathbf{e}_{-1}(u_j^{(n)} - u_k^{(n-1)}) \prod_{k \neq j}^{M^{(n)}} \mathbf{e}_2(u_j^{(n)} - u_k^{(n)}) \prod_{k=1}^{M^{(n+1)}} \mathbf{e}_{-1}(u_j^{(n)} - u_k^{(n+1)}) = \\
& = \begin{cases} \prod_{l=1}^L \mathbf{e}_1(u_j^{(1)} - a_l - i) & , \quad n = 1 \\ 1 & , \quad 1 < n < \mathcal{M} \end{cases}, \quad 1 \leq j \leq M^{(n)}, \tag{6.2}
\end{aligned}$$

$$\begin{aligned}
& \prod_{k=1}^{M^{(n-1)}} \mathbf{e}_1(u_j^{(n)} - u_k^{(n-1)}) \prod_{k \neq j}^{M^{(n)}} \mathbf{e}_{-2}(u_j^{(n)} - u_k^{(n)}) \prod_{k=1}^{M^{(n+1)}} \mathbf{e}_1(u_j^{(n)} - u_k^{(n+1)}) = 1, \\
& 1 \leq j \leq M^{(n)}, \quad \mathcal{M} < n \leq \mathcal{M} + \mathcal{N} - 1, \tag{6.3}
\end{aligned}$$

$$\prod_{k=1}^{M^{(\mathcal{M}-1)}} \mathbf{e}_{-1}(u_j^{(\mathcal{M})} - u_k^{(\mathcal{M}-1)}) \prod_{k=1}^{M^{(\mathcal{M}+1)}} \mathbf{e}_1(u_j^{(\mathcal{M})} - u_k^{(\mathcal{M}+1)}) = 1, \quad 1 \leq j \leq M^{(\mathcal{M})}. \tag{6.4}$$

Since here  $T_{an}(u) = R_{an}(u)$ , its value at  $u = 0$  is proportional to the graded permutation operator between the  $a$  (auxiliary) and  $n$  (quantum) spaces. Thus, we can construct a local Hamiltonian in the usual way. Choosing  $a_n = 0$  for all sites, we get

$$H = -i \frac{d}{du} (\ln st(u)) \Big|_{u=0} = - \sum_{n=1}^L P_{n-1,n} \quad \text{with} \quad P_{01} = P_{L1}. \quad (6.5)$$

Here  $P_{n-1,n}$  is the graded permutation between sites  $n-1$  and  $n$ . In particular, in the  $\mathcal{M} = 1$ ,  $\mathcal{N} = 2$  case we recover the supersymmetric  $t - J$  model, which corresponds to the  $\mathcal{Y}(1|2)$  case [18]. The energies corresponding to the Hamiltonian (6.6) can be calculated by taking the logarithmic derivative of  $\Lambda(u)$  and evaluating at  $u = 0$ , and are given by

$$E = L + \sum_{j=1}^{M^{(1)}} \frac{1}{(u_j^{(1)})^2 + \frac{1}{4}},$$

where the Bethe parameters  $u_j^{(n)}$  are solution to the Bethe equations (6.2-6.4) with  $a_n = 0$ ,  $\forall n$ .

A slightly generalized case is obtained taking  $a_p = a \neq 0$  for a particular site  $p$ , and  $a_n = 0$  for  $n \neq p$ . This leads to the following Hamiltonian:

$$H = - \sum_{\substack{n=1 \\ n \neq p, p+1}}^L P_{n-1,n} + \frac{1}{a^2 + 1} (a^2 P_{p-1,p+1} + P_{p+1,p} - i a P_{p+1,p-1} P_{p,p-1} + i a P_{p,p-1} P_{p+1,p-1}).$$

The energies get modified as follows:

$$E = -L + \frac{a}{a+i} + \sum_{j=1}^{M^{(1)}} \frac{1}{(u_j^{(1)})^2 + \frac{1}{4}}$$

for the where the Bethe parameters  $u_j^{(n)}$  are solution to the Bethe equations (6.2-6.4), with now inhomogeneities  $a_n = \delta_{n,p} a$ .

## 6.2 Closed spin chain with an impurity

Another case to which our formalism easily applies is the super-spin chain with one site (the so-called impurity) in a representation different from the others. The easiest case is again the spin chain where all sites are in the fundamental representation except for the  $p^{th}$ , associated to the highest weight  $\mu_i^{[p]}$ ,  $i = 1, \dots, \mathcal{M} + \mathcal{N}$ . The right hand sides of the Bethe equations are modified as follows:

$$\frac{\lambda_n(u_j^{(n)} - i \frac{n}{2})}{\lambda_{n+1}(u_j^{(n)} - i \frac{n}{2})} = \begin{cases} \left( \epsilon_1(u_j^{(1)} - i) \right)^{L-1} \frac{u_j^{(1)} - \frac{i}{2} - i\mu_1^{[p]}}{u_j^{(1)} - \frac{i}{2} - i\mu_2^{[p]}} & , \quad n = 1, \\ \frac{u_j^{(n)} - i \frac{n}{2} - i\mu_n^{[p]}}{u_j^{(n)} - i \frac{n}{2} - i\mu_{n+1}^{[p]}} & , \quad 1 < n < \mathcal{M}, \end{cases} \quad (6.6)$$

$$\frac{\lambda_n(u_j^{(n)} - i(\mathcal{M} - \frac{n}{2}))}{\lambda_{n+1}(u_j^{(n)} - i(\mathcal{M} - \frac{n}{2}))} = \frac{u_j^{(n)} - i(\mathcal{M} - \frac{n}{2}) - i\mu_n^{[p]}}{u_j^{(n)} - i(\mathcal{M} - \frac{n}{2}) - i\mu_{n+1}^{[p]}}, \quad \mathcal{M} < n \leq \mathcal{M} + \mathcal{N} - 1, \quad (6.7)$$

$$\frac{\lambda_{\mathcal{M}+1}(u_j^{(\mathcal{M})} - i\frac{\mathcal{M}}{2})}{\lambda_{\mathcal{M}}(u_j^{(\mathcal{M})} - i\frac{\mathcal{M}}{2})} = \frac{u_j^{(\mathcal{M})} - i\frac{\mathcal{M}}{2} + i\mu_{\mathcal{M}+1}^{[p]}}{u_j^{(\mathcal{M})} - i\frac{\mathcal{M}}{2} - i\mu_{\mathcal{M}}^{[p]}}, \quad (6.8)$$

where we set again  $a_n = 0$  for all sites. The transfer matrix and the Hamiltonian of the  $L$ -sites spin chain with one impurity can be written as

$$st(u) = str_a (R_{a,1}(u) \cdots R_{a,p-1}(u) T_{a,p}(u) R_{a,p+1}(u) \cdots R_{a,L}(u)), \quad (6.9)$$

$$H = -i T_{p+1,p}^{-1}(0) - P_{p-1,p+1} T_{p-1,p}^{-1}(0) T_{p+1,p}(0) - \sum_{n=1, n \neq p-1, p}^L P_{n,n+1}. \quad (6.10)$$

It is worth noticing that all the quantum spaces  $n$  (but the  $p$ -th one) are isomorphic to the auxiliary space  $a$ . Hence,  $T_{n,p}(u)$ ,  $n \neq p$ , is defined in the same way  $T_{a,p}(u)$  was introduced. The spectrum of the Hamiltonian (6.10) is then given by:

$$E = -(L-1) + i \frac{\mu_1'(0)}{\mu_1(0)} + \sum_{j=1}^{M^{(1)}} \frac{1}{(u_j^{(1)})^2 + \frac{1}{4}}.$$

### 6.3 Closed alternating spin chains

In alternating spin chains, the spins along the chain belong alternatively to two different representations. As a particular example, one can take an even number of sites  $L$  for the chain, and let the spins in the even sites be in the fundamental representation, while the spins in the odd sites are in a different one. The transfer matrix for such a chain will then be given by

$$st(u) = str_a (T_{a,1}(u) R_{a,2}(u) \cdots T_{a,L-1}(u) R_{a,L}(u)),$$

here the auxiliary space  $a$  is  $\mathcal{M} + \mathcal{N}$  dimensional. One gets a local Hamiltonian

$$H = -i \frac{d}{du} (\ln st(u)) \Big|_{u=0} = - \sum_{j=1}^{L/2} T_{2j-2,2j-1}^{-1}(0) \left\{ i \mathbb{I} + P_{2j-2,2j} T_{2j-2,2j-1}(0) \right\}. \quad (6.11)$$

Denoting by  $\mu_j$ ,  $j = 1, \dots, \mathcal{M} + \mathcal{N}$  the weights of the representation on odd sites, and  $\mu_j(u) = u - i(-1)^{[j]} \mu_j$ , one gets for the eigenvalues (3.12)

$$\lambda_k(u) = \begin{cases} (u-i)^{L/2} \mu_1(u)^{L/2} & k = 1, \\ u^{L/2} \mu_k(u)^{L/2} & 1 < k \leq \mathcal{M} + \mathcal{N}. \end{cases}$$

where we set  $a_n = 0$  for all  $n$ . This leads to the spectrum

$$E = -\frac{L}{2} \left( 1 - i \frac{\mu_1'(0)}{\mu_1(0)} \right) + \sum_{j=1}^{M^{(1)}} \frac{1}{(u_j^{(1)})^2 + \frac{1}{4}},$$

### 6.3.1 Specialization to fundamental–adjoint alternating spin chain

Choosing e.g. the adjoint representation for the odd sites, i.e. highest weights  $\mu_i^{[n]} = \delta_{i,1}$  for even  $n$  and  $\mu_i^{[n]} = \delta_{i,1} + \delta_{i,\mathcal{M}+\mathcal{N}}$  for odd  $n$ , one gets the following form for the eigenvalues

$$\lambda_k(u) = \begin{cases} (u-i)^L & k=1, \\ u^L & 1 < k < \mathcal{M} + \mathcal{N}, \\ (u+i)^{L/2} u^{L/2}, & k = \mathcal{M} + \mathcal{N}, \end{cases}$$

The Bethe equations for  $1 \leq n \leq \mathcal{M}$  remain as in the fundamental representation case (6.2) and (6.4), while the equations (6.3) for  $\mathcal{M} < n \leq \mathcal{M} + \mathcal{N} - 1$  are modified as follows:

$$\begin{aligned} & \prod_{k=1}^{M^{(n-1)}} \mathbf{e}_1(u_j^{(n)} - u_k^{(n-1)}) \prod_{k \neq j}^{M^{(n)}} \mathbf{e}_{-2}(u_j^{(n)} - u_k^{(n)}) \prod_{k=1}^{M^{(n+1)}} \mathbf{e}_1(u_j^{(n)} - u_k^{(n+1)}) = \\ & = \begin{cases} 1 & , \mathcal{M} < n < \mathcal{M} + \mathcal{N} - 1, \\ \left( \mathbf{e}_{-1}(u_j^{(n)} - i \frac{\mathcal{M}-\mathcal{N}}{2}) \right)^{L/2} & , n = \mathcal{M} + \mathcal{N} - 1, \end{cases} \end{aligned} \quad (6.12)$$

with  $1 \leq j \leq M^{(n)}$ . In this case, the monodromy matrix  $T_{aj}(u)$  can be obtained through the usual fusion procedure [28], starting with the fused  $R$  matrix:

$$R_{a(bc)}(u) = \mathcal{P}_{bc}^+ R_{ac}(-u) R_{ab}^{t_b}(u) \mathcal{P}_{bc}^+, \quad (6.13)$$

where  $\mathcal{P}_{bc}^+ = \mathbb{I}_{bc} - \frac{1}{2\rho} Q_{bc}$  is a projector of dimension  $\mathcal{M} + \mathcal{N} - 1$ . The tensor product of the spaces  $b$  and  $c$  is then considered as a single quantum space, and  $T_{aj}(u)$  is obtained from  $R_{a(bc)}$  through a suitable similarity transformation applied on both sides of (6.13), yielding:

$$\begin{aligned} R_{aj}(u) &= u \mathbb{I}_{aj} + i (\mathbf{e}_a \cdot \mathbf{e}_j), \\ T_{aj}(u) &= u \mathbb{I}_{aj} - i (\mathbf{e}_a \cdot \mathcal{E}_j), \end{aligned}$$

where  $\mathbf{e}$  and  $\mathcal{E}$  respectively denote the  $gl(\mathcal{M}|\mathcal{N})$  generators in the fundamental and adjoint representations. The inner product  $\cdot$  is defined, as usual, by means of the invariant, nondegenerate bilinear form  $K^{\alpha\beta}$  on  $gl(\mathcal{M}|\mathcal{N})$ , which is given as the supertrace on two generators  $K_{\alpha\beta} = \text{str}(\mathcal{E}_\alpha \mathcal{E}_\beta)$ :

$$\mathbf{A} \cdot \mathbf{B} = \sum_{\alpha, \beta} (K^{-1})^{\alpha\beta} A_\alpha A_\beta$$

Fusion allows also a direct calculation of  $T_{aj}(u)^{-1}$ , so that one gets an explicit expression of the Hamiltonian (6.11). It involves nearest–neighbour and next–nearest–neighbour interaction terms:

$$H = \sum_{j=1, j \text{ even}}^{L/2} H_{j,j+1}^{(1)} + \sum_{j=1, j \text{ odd}}^{L/2} H_{j-1,j,j+1}^{(2)}, \quad (6.14)$$

where

$$H_{j,j+1}^{(1)} = -\mathbf{e}_j \cdot \mathcal{E}_{j+1} + \frac{1}{2\rho} (\mathbf{e}_j \cdot \mathcal{E}_{j+1})^2, \quad \rho = (\mathcal{M} - \mathcal{N})/2 \quad (6.15)$$

$$H_{j-1,j,j+1}^{(2)} = \frac{1}{2\rho} (\mathbf{e}_{j-1} \cdot \mathcal{E}_j) \{2\rho + (\mathbf{e}_{j-1} \cdot \mathcal{E}_j)\} (\mathbf{e}_{j-1} \cdot \mathbf{e}_{j+1}) (\mathbf{e}_{j-1} \cdot \mathcal{E}_j). \quad (6.16)$$

The spectrum of the Hamiltonian (6.14) reads:

$$E = -L + \sum_{j=1}^{M^{(1)}} \frac{1}{(u_j^{(1)})^2 + \frac{1}{4}}.$$

## 6.4 The open alternating spin chain

We define the transfer matrix for a  $2L$ -site open alternating chain as:

$$\begin{aligned} b(u) = & \text{str}_a \left( K^+(u) T_{a,1}(u) R_{a,2}(u) \cdots T_{a,2L-1}(u) R_{a,2L}(u) K(u) \times \right. \\ & \left. \times R_{a,2L}^{-1}(-u) T_{a,2L-1}^{-1}(-u) \cdots R_{a,2}^{-1}(-u) T_{a,1}^{-1}(-u) \right) \end{aligned}$$

Here the matrices acting on the even sites are in the fundamental representation, coinciding again with  $R(u)$ , and the ones for the non-fundamental are denoted with  $T(u)$  and act on the odd sites of the chain. A local Hamiltonian can be obtained by taking the derivative of  $b(u)$ :

$$H = \frac{1}{\xi \xi' \rho} \left. \frac{d}{du} b(u) \right|_{u=0},$$

where we remind  $\rho = \mathcal{M} - \mathcal{N}$  while  $\xi$  and  $\xi'$  characterize the boundary matrices  $K(u)$  and  $K^+(u)$  respectively as in (2.83), (5.3). One shows that

$$\begin{aligned} H = & \frac{1}{\xi} K'_{2l}(0) + \frac{1}{\xi' \rho} \text{str}_a \left( \frac{d K_a^+(u)}{du} \right) \Big|_{u=0} + \frac{2}{\rho} \text{str}_a \{ (i\mathbb{I} + T_{a,1}(0) P_{a2}) T_{a,1}^{-1}(0) \} \\ & + 2 \sum_{k=2}^l (i\mathbb{I} + T_{2k-2,2k-1}(0) P_{2k-2,2k}) T_{2k-2,2k-1}^{-1}(0). \end{aligned}$$

We will suppose that the gradation is such that  $c_m \neq 0$  for  $m > 0$ . In the case of distinguished gradation, this amounts to choose  $\mathcal{M} > \mathcal{N}$ . Then, the energy spectrum is given by:

$$\begin{aligned} E = & \beta L \left( 1 + \sum_{m=1}^{\mathcal{M}+\mathcal{N}-1} \frac{1}{c_m} - i \sum_{m=1}^{\mathcal{M}+\mathcal{N}} \frac{\mu'_m(-ic_{m-1})}{\mu_m(-ic_{m-1})} \right) - \sum_{j=1}^{M^{(1)}} \frac{2\beta}{(u_j^{(1)})^2 + \frac{1}{4}} \\ & + i\beta \frac{\xi + \xi'}{\xi \xi'} + 2\beta \frac{1 - \rho}{\rho}, \end{aligned} \tag{6.17}$$

where  $\beta = \beta_1(0)$ . For the distinguished Dynkin diagram, and choosing the adjoint representation for the odd sites, the Bethe equations read, for  $1 \leq n \leq M^{(l)}$  with  $1 \leq l \leq \mathcal{M} + \mathcal{N} - 1$ :

$$\begin{aligned} & \prod_{j=1}^{M^{(1)}} \mathbf{e}_2(u_n^{(1)} - u_j^{(1)}) \mathbf{e}_2(u_n^{(1)} + u_j^{(1)}) \prod_{j=1}^{M^{(2)}} \mathbf{e}_{-1}(u_n^{(1)} - u_j^{(2)}) \mathbf{e}_{-1}(u_n^{(1)} + u_j^{(2)}) = \\ & = - \left( \mathbf{e}_{-1}(u_n^{(1)} - i) \mathbf{e}_{-3}(u_n^{(1)} + i) \right)^L \mathbf{e}_1(u_n^{(1)}) Q_1(u_n^{(1)} - \frac{i}{2}), \end{aligned}$$

$$\begin{aligned}
& \prod_{j=1}^{M^{(l)}} \mathbf{e}_2(u_k^{(l)} - u_j^{(l)}) \mathbf{e}_2(u_k^{(l)} + u_j^{(l)}) \prod_{\tau=\pm 1} \prod_{j=1}^{M^{(l+\tau)}} \mathbf{e}_{-1}(u_k^{(l)} - u_j^{(l+\tau)}) \mathbf{e}_{-1}(u_k^{(l)} + u_j^{(l+\tau)}) = \\
& = -\mathbf{e}_1(u_n^{(l)}) Q_l(u_n^{(l)} - \frac{i l}{2}), \quad 1 < l < \mathcal{M},
\end{aligned}$$

$$\begin{aligned}
& \prod_{j=1}^{M^{(\mathcal{M}+1)}} \mathbf{e}_1(u_n^{(\mathcal{M})} - u_j^{(\mathcal{M}+1)}) \mathbf{e}_1(u_n^{(\mathcal{M})} + u_j^{(\mathcal{M}+1)}) \prod_{j=1}^{M^{(\mathcal{M}-1)}} \mathbf{e}_{-1}(u_n^{(\mathcal{M})} - u_j^{(\mathcal{M}-1)}) \mathbf{e}_{-1}(u_n^{(\mathcal{M})} + u_j^{(\mathcal{M}-1)}) = \\
& = Q_{\mathcal{M}}(u_n^{(\mathcal{M})} - \frac{i \mathcal{M}}{2}),
\end{aligned}$$

$$\begin{aligned}
& \prod_{j=1}^{M^{(l)}} \mathbf{e}_{-2}(u_k^{(l)} - u_j^{(l)}) \mathbf{e}_{-2}(u_k^{(l)} + u_j^{(l)}) \prod_{\tau=\pm 1} \prod_{j=1}^{M^{(l+\tau)}} \mathbf{e}_1(u_k^{(l)} - u_j^{(l+\tau)}) \mathbf{e}_1(u_k^{(l)} + u_j^{(l+\tau)}) = \\
& = -\mathbf{e}_{-1}(u_n^{(l)}) Q_l(u_n^{(l)} - \frac{i}{2}(2\mathcal{M} - l)), \quad \mathcal{M} < l < \mathcal{M} + \mathcal{N} - 1,
\end{aligned}$$

$$\begin{aligned}
& \prod_{j=1}^{M^{(l)}} \mathbf{e}_{-2}(u_k^{(l)} - u_j^{(l)}) \mathbf{e}_{-2}(u_k^{(l)} + u_j^{(l-1)}) \prod_{\tau=\pm 1} \prod_{j=1}^{M^{(l+\tau)}} \mathbf{e}_1(u_k^{(l)} - u_j^{(l+\tau)}) \mathbf{e}_1(u_k^{(l)} + u_j^{(l+\tau)}) = \\
& = -\left( \mathbf{e}_{-1}(u_n^{(l)} - i\rho) \mathbf{e}_{-1}(u_n^{(l)} + i\rho) \right)^L \mathbf{e}_1(u_n^{(l)}) Q_{\mathcal{M}+\mathcal{N}-1}(u_n^{(l)} - i(\rho - \frac{1}{2})), \\
& l = \mathcal{M} + \mathcal{N} - 1.
\end{aligned}$$

In the above equations, we set

$$Q_l(u) = \frac{K_l(u) K_l^+(u)}{K_{l+1}(u) K_{l+1}^+(u)},$$

according to the chosen boundary matrices.

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